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Abstract

We revisit the one-dimensional one-phase Stefan problem with a Dirichlet boundary condition at $x = 0$ as stated in the book of Avner Friedman about parabolic equations [F3]. We prove that under rather general hypotheses on the initial data, the solution converges to a self-similar profile as $t \to +\infty$.

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1 Introduction

In this article, we revisit a standard one-dimensional one-phase Stefan problem. This free boundary problem arises in very simple physical situations and has been studied by numerous authors; in particular we should mention a chapter of the book of Avner Friedman on parabolic equations

(Chapter 8 of [F3, p.215]). This problem is given by

$$\begin{cases} u_t = u_{xx}, & t>0, 0<x<s(t), \\ u(0,t)=h, & t>0, \\ u(s(t),t)=0, & t>0, \\ \dfrac{ds(t)}{dt} = -u_x(s(t),t), & t>0, \\ s(0)=b_0, & \\ u(x,0)=u_0(x), & 0<x<b_0 \end{cases} \tag{1.1}$$

where $x=s(t)$ is the unknown free boundary which is to be found together with $u(x,t)$.

Friedman [F3] proves that this problem has a unique smooth classical solution $\big(u(x,t),s(t)\big)$ in $Q:=\{(x,t), t>0, 0<x<s(t)\}$. Moreover it follows from Schaeffer [S] and Friedman [F1] that $s\in C^\infty(0,\infty)$ and that u is infinitely differentiable up to the free boundary s.

The purpose of this paper is to study the large time behavior of the solution pair (u,s). Also let us mention some previous results from literature. Meirmanov [M] has proved that $\dfrac{s(t)}{\sqrt{t}} \to a$, where a is the unique solution of the nonlinear equation (1.3) below. Also, Ricci and Xie [R] have performed a stability analysis of some special solutions of a related one-phase Stefan problem on the semi-infinite interval $(s(t),\infty)$. In particular, they mention that the interface $s(t)$ behaves as $\beta\sqrt{t}$ for some positive constant β which they characterize. Moreover, Aiki and Muntean [AM1, AM2], as mentioned in [Z], have proved the existence of two positive constants c and C independent of t such that

$$c\sqrt{t} \leqslant s(t) \leqslant C\sqrt{t+1} \ \text{ for all } \ t\geqslant 0,$$

in the case of a more complicated system.

In this article, we will prove that the solution pair (u,s) converges to a self-similar solution as $t\to\infty$.

First, let us define the self-similar solution. To do so, we introduce the self-similar variable $\eta = \dfrac{x}{\sqrt{t+1}}$. Then, the self-similar solution is given by

$$u(x,t) = U\left(\frac{x}{\sqrt{t+1}}\right) = U(\eta) = h\left[1-\frac{\int_0^\eta e^{-\frac{s^2}{4}}ds}{\int_0^a e^{-\frac{s^2}{4}}ds}\right] \ \text{ for all } \eta\in(0,a), \tag{1.2}$$

where a is characterized as the unique solution of the nonlinear equation

$$h = \frac{a}{2}e^{\frac{a^2}{4}}\int_0^a e^{-\frac{s^2}{4}}ds. \tag{1.3}$$

In the first step, we will write the problem (1.1) in terms of η and t. To do so, we set

$$\begin{cases} V(\eta,t) = u(x,t), \\ a(t) = \dfrac{s(t)}{\sqrt{t+1}}. \end{cases} \tag{1.4}$$

However, the partial differential equation for V which we obtain explicitly involves the time variable t. It is given by

$$(t+1)V_t = V_{\eta\eta} + \frac{\eta}{2}V_\eta, \quad t>0, \quad 0<\eta<a(t). \tag{1.5}$$

This leads us to perform the change of time variable $\tau = \ln(t+1)$. A similar change of variables was performed by [HH]. The full time evolution problem corresponding to the system (1.1) in coordinates η and τ is given by

$$\begin{cases} W_\tau = W_{\eta\eta} + \dfrac{\eta}{2} W_\eta, & \tau > 0, \quad 0 < \eta < b(\tau), \\ W(0,\tau) = h, & \tau > 0, \\ W(b(\tau),\tau) = 0, & \tau > 0, \\ \dfrac{db(\tau)}{d\tau} + \dfrac{b(\tau)}{2} = -W_\eta(b(\tau),\tau), & \tau > 0, \\ b(0) = b_0, & \\ W(\eta,0) = u_0(\eta), & 0 < \eta < b_0 \end{cases} \tag{1.6}$$

where $b(\tau) = a(t)$. We shall denote by $(W(\eta,\tau,(u_0,b_0)), b(\tau,(u_0,b_0)))$ the solution pair of (1.6) with the initial conditions (u_0, b_0).
It is in the coordinates η and τ that we will rigorously characterize the large time behavior of the solution pair (W,b). However, for technical reasons, we sometimes have to use different variables, namely (y,τ) with $y = \dfrac{\eta}{b(\tau)}$ for all $0 < \eta < b(\tau)$. The problem is then transformed into a problem on a fixed domain.

Organization of the paper. In Section 2, we introduce the Stefan problem [F2] and recall known well-posedness and regularity results [F1, S]. Using a maximum principle [F3], we show that if u_0 is nonnegative and bounded then the solution u is also nonnegative and bounded.

In Section 3, we start by defining a notion of upper and lower solutions for Problem (1.1). Then, we prove a comparison principle in the (x,t) coordinates for a pair of upper and lower solutions of Problem (1.1).

In Section 4, we construct the self-similar solution (U,a). We will show that U is as given by (1.2) and a is characterized as the unique solution of the nonlinear equation (1.3).

In Section 5, we transform Problem (1.1) in coordinates (x,t) to obtain an equivalent problem, Problem (1.6), in coordinates (η,τ) where the solution pair becomes (W,b). We present an equivalent comparison principle in these coordinates and a class of functions which include both the lower and upper-solutions. We use the notation $(\bar{\mathcal{W}}, \bar{b})$ for the upper-solution, respectively $(\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)$ for the lower-solution depending on a parameter $\lambda \geqslant 0$, and we construct a function (W_λ, b_λ) such that

$$(W_\lambda, b_\lambda) \text{ is } \begin{cases} \text{an upper solution} & \text{if } 0 \leqslant \lambda \leqslant 1, \\ \text{a lower solution} & \text{if } \lambda \geqslant 1. \end{cases} \tag{1.7}$$

Then, we prove the monotonicity in time of the solution pair (W,b) of the time evolution Problem (1.6) with the two initial conditions $(\bar{\mathcal{W}}, \bar{b})$ and $(\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)$. In other words, we show that starting from a lower solution, the solution $\underline{W}(\eta,\tau) := W\big(\eta,\tau,(\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)\big)$ increases in time as $\tau \to \infty$ to a limit function ψ and the corresponding moving boundary $\underline{b}(\tau) := b\big(\tau,(\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)\big)$ increases to a limit $\underline{b}_\infty$. Similarly, one can show that starting from an upper solution, the solution decreases to another limit function ϕ as $\tau \to \infty$ and the moving boundary $\bar{b}$ converges to a limit $\bar{b}_\infty$.
At the end of this section, we discuss some properties of upper and lower solutions to conclude that they are ordered functions.
However, we do not know yet whether ψ and ϕ coincide with the self-similar profile U and whether $\underline{b}_\infty$ and $\bar{b}_\infty$ coincide with the point a. In order to prove these results we first have to show extra a priori estimates which we do in the following section.

In Section 6, we prove a number of a priori estimates some in the moving domain and some in the fixed domain. Indeed, we temporarily pass to fixed domain $(y,\tau) \in (0,1) \times \mathbb{R}^+$ to avoid technical problems related to the characterization of the limits $\underline{b}_\infty$ and $\bar{b}_\infty$. In other words, we need to show that $\underline{W}_\eta(\underline{b}(\tau),\tau)$ converges to $\psi_\eta(\underline{b}_\infty)$ as $\tau \to \infty$. This requests to prove the uniform convergence of $\underline{W}_\eta(\eta,\tau)$ to its limit as $\tau \to \infty$ which we can more easily do in the fix domain coordinates. Section 7 is devoted to the study of the limits as $\tau \to \infty$. More precisely, we prove that $(\psi, \underline{b}_\infty)$ verifies the following conditions

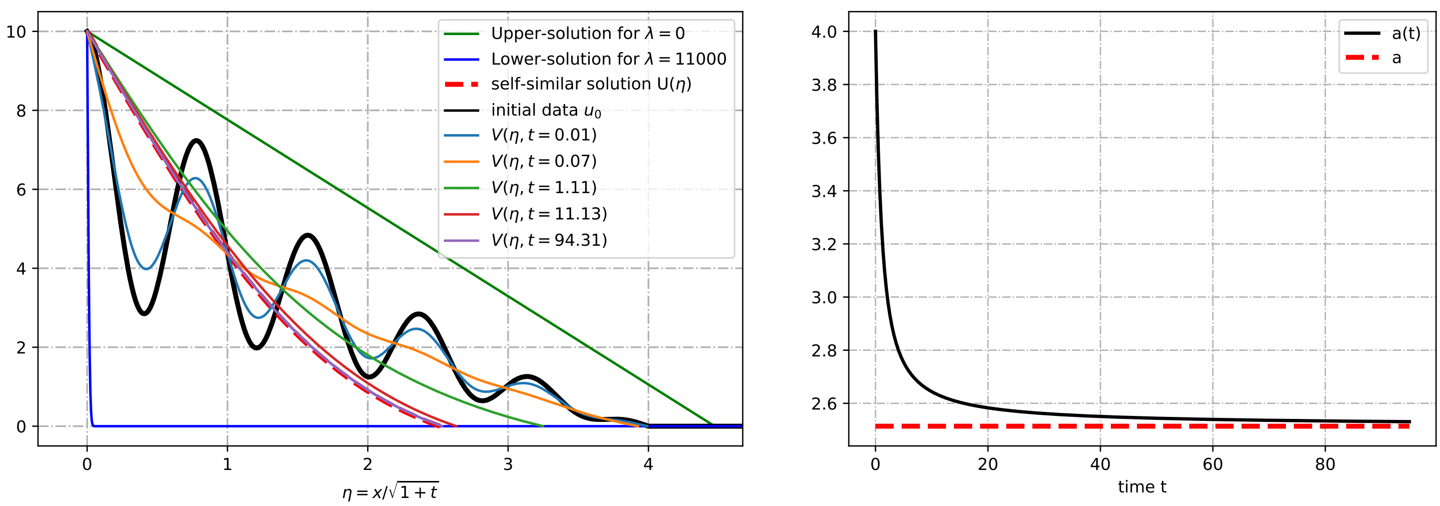
$$\psi(0) = h, \quad \psi(\underline{b}_\infty) = 0, \quad \frac{\underline{b}_\infty}{2} = -\psi_\eta(\underline{b}_\infty).$$

and ψ satisfies the ordinary differential equation

$$\psi_{\eta\eta} + \frac{\eta}{2}\psi_\eta = 0.$$

Similarly, it turns out that $\Big(W\big(\eta,\tau,(\bar{\mathcal{W}},\bar{b})\big), b\big(\tau,(\bar{\mathcal{W}},\bar{b})\big)\Big)$ converges as $\tau \to \infty$ towards the unique solution $(\phi, \bar{b}_\infty)$ of the stationary problem corresponding to Problem (1.6). At the end of Section 7, we show that the solution pair $(\psi, \underline{b}_\infty)$ coincides with the unique solution (U, a) of Problem (4.4) which coincides also with the solution pair $(\phi, \bar{b}_\infty)$.

Next, we present the results of some numerical simulations. We choose the initial data (u_0, b_0) such that $\underline{b}_\lambda \leqslant b_0 \leqslant \bar{b}$ and $\underline{\mathcal{W}}_\lambda \leqslant u_0 \leqslant \bar{\mathcal{W}}$. Figure 1 shows the large behavior of the solution pair (V, a) defined in (1.4).



(a) Time evolution of the unknown function $V(\eta, t)$. (b) Time evolution of the moving boundary $a(t)$.

Figure 1: Large time behavior of the solution pair (V, a).

To state an exact formulation of the results of this article, it is most convenient to use the variable

y lying in $[0,1]$. In the variables (y,η), the problem for $\big(\hat{W}(y,\tau),b(\tau)\big)=\big(W(\eta,\tau),b(\tau)\big)$ is given by

$$\begin{cases} \hat{W}_\tau(y,\tau)=\dfrac{1}{b^2(\tau)}\hat{W}_{yy}(y,\tau)+y\left(\dfrac{d\,ln\big(b(\tau)\big)}{d\tau}+\dfrac{1}{2}\right)\hat{W}_y(y,\tau), & \tau>0,\ \ 0<y<1,\\ \hat{W}(0,\tau)=h, & \tau>0,\\ \hat{W}(1,\tau)=0, & \tau>0,\\ \dfrac{1}{2}\dfrac{db^2(\tau)}{d\tau}+\dfrac{b^2(\tau)}{2}=-\hat{W}_y(1,\tau), & \tau>0,\\ b(0)=b_0,\\ \hat{W}(y,0)=u_0(b_0y), & 0\leqslant y\leqslant 1. \end{cases} \tag{1.8}$$

The main result of this article is the following. We suppose that the initial data (u_0,b_0) satisfies the hypothesis:
$\mathbf{H}_0$: $b_0\leqslant\bar{b}$ and $u_0\in\mathbb{W}^{1,\infty}(0,b_0)$ with $u_0(0)=h$ and

$$\begin{aligned} &u_0(x)=0 \ \text{ for all } x\geqslant b_0,\\ &0\leqslant u_0(x)\leqslant h\big(1-\frac{x}{\sqrt{2h}}\big) \ \text{ for all } 0\leqslant x\leqslant b_0. \end{aligned}$$

Main Theorem 1.1. Suppose that (u_0,b_0) satisfies the hypothesis $\mathbf{H}_0$. The unique solution $(\hat{W},b)$ of Problem (1.8) is such that

$$\lim_{\tau\to+\infty}||\hat{W}(.,\tau)-\hat{U}||_{C([0,1])}=0, \tag{1.9}$$

$$\lim_{\tau\to+\infty}b(\tau)=a, \tag{1.10}$$

where $(\hat{U},a)$ is the unique solution of the stationary problem

$$\begin{cases} \dfrac{1}{a^2}\hat{U}_{yy}+\dfrac{y}{2}\hat{U}_y=0, & 0<y<1,\\ \hat{U}(0)=h,\quad \hat{U}(1)=0,\\ \dfrac{a^2}{2}=-\hat{U}_y(1) \end{cases} \tag{1.11}$$

which is equivalent to the stationary problem corresponding to Problem (1.6)

$$\begin{cases} U_{\eta\eta}+\dfrac{\eta}{2}U_\eta=0, & 0<\eta<a,\\ U(0)=h,\quad U(a)=0,\\ \dfrac{a}{2}=-U_\eta(a), \end{cases} \tag{1.12}$$

for the self-similar solution of Problem (1.1).

Remark 1.2. (1.10) is equivalent to the convergence result

$$\frac{s(t)}{\sqrt{t+1}}\to a \ \text{ as } \ t\to+\infty, \tag{1.13}$$

which was already proved by Meirmanov [M].

2 Friedman's formulation

Let $h > 0$, $b > 0$. We define the function space

$$X^h(b) := \{u_0(x) \in C[0,\infty), \;\; u_0(0) = h, \; u_0(x) \geqslant 0 \text{ for } 0 \leqslant x < b, \; u_0(x) = 0 \text{ for } x \geqslant b\}.$$

and we consider the problem

$$\begin{cases} u_t = u_{xx}, & t > 0, 0 < x < s(t), \\ u(0,t) = h, & t > 0, \\ u(s(t),t) = 0, & t > 0, \\ \dfrac{ds(t)}{dt} = -u_x(s(t),t), & t > 0, \\ s(0) = b_0, & \\ u(x,0) = u_0(x) \in X^h(b_0). & \end{cases} \tag{2.1}$$

Problem (2.1) is a free boundary problem where $x = s(t)$ is the free boundary to be found together with the unknown function $u(x,t)$.

Definition 2.1. Let $T > 0$. We say that the pair (u, s) is a classical solution of Problem (2.1) if

(i) $s(t)$ is continuously differentiable for $0 \leqslant t \leqslant T$;

(ii) $u \in C(\overline{Q_T})$, where $Q_T := \{(x,t), t \in (0,T], 0 < x < s(t)\}$;

(iii) $u \in C^{2,1}(Q_T)$;

(iv) $u_x \in C(\overline{Q_T^\delta})$ for all $\delta > 0$ where $Q_T^\delta = \{(x,t), t \in (\delta,T], 0 < x < s(t)\}$;

(v) the equations of Problem (2.1) are satisfied.

Let $(u(x,t), s(t))$ be a solution of (2.1) for all $0 \leqslant t \leqslant T$. We extend u by:

$$u(x,t) = 0 \text{ for } x \geqslant s(t), \tag{2.2}$$

so that $u(\cdot,t)$ is defined for all $x \geqslant 0$.

Theorem 2.2 ([F2, Theorem 1])**.** Let $h > 0, b > 0$ and $u_0 \in X^h(b)$. Then, there exists a unique solution $(u(x,t), s(t))$ of (2.1) for all $t > 0$ in the classical sense. Moreover, the solution (u, s) is such that s is infinitely differentiable on $(0,\infty)$ and u is infinitely differentiable up to the free boundary for all $t > 0$ [F1],[S]. Furthermore, the function $s(t)$ is strictly increasing in t.

Proposition 2.3. Let $h > 0, b > 0$ and $u_0 \in X^h(b)$ such that $0 \leqslant u_0 \leqslant h$. Then, the solution $(u(x,t), s(t))$ of (2.1) is such that $0 \leqslant u(x,t) \leqslant h$ for all $(x,t) \in Q_T$.

Proof. We apply the strong maximum principle (Theorem 1 of [F3, p.34]) which states that if u attains its minimum or its maximum in an interior point $(x^0, t^0) \in Q_T$, then u is constant in Q_{t^0}. However, since $u(0,t) = h > 0$ for $t \in (0,T]$ and $u\big(s(t),t\big) = 0$, $u(.,t)$ cannot be constant in space on $\big(0, s(t)\big)$, so that u attains its minimum and its maximum on the boundary $\Gamma := \{(0,t), 0 \leqslant t \leqslant T\} \cup \{(x,0), 0 < x < b\} \cup \{(s(t),t), 0 \leqslant t \leqslant T\}$. As $0 \leqslant u_0 \leqslant h$, we conclude that $0 \leqslant u(x,t) \leqslant h$ for all $(x,t) \in Q_T$. □

3 Comparison principle

To begin with, we define a notion of lower and upper solutions.

Definition 3.1. For $u \in C(\overline{Q_T}) \cap C^{2,1}(Q_T)$, we define $\mathcal{L}(u) := u_t - u_{xx}$.
$(\underline{u}, \underline{s})$ is a lower solution of the Problem (2.1) if it satisfies

$$\begin{cases} \mathcal{L}(\underline{u}) = \underline{u}_t - \underline{u}_{xx} \leqslant 0 \ \text{ in } \mathrm{Q_T}, \\ \underline{u}(0,t) \leqslant h, \quad \underline{u}(\underline{s}(t),t) = 0, & t > 0, \\ \dfrac{d\underline{s}(t)}{dt} = -\underline{u}_x(\underline{s}(t),t), & t > 0, \\ \underline{s}(0) \leqslant b_0, \\ \underline{u}(x,0) \leqslant u_0(x), & x \in (0, b_0). \end{cases} \tag{3.1}$$

$(\bar{u}, \bar{s})$ is an upper solution of the Problem (2.1) if it satisfies (3.1) with all $\leqslant$ replaced by $\geqslant$.

Theorem 3.2 (Comparison principle)**.** Let $(u_1(x,t), s_1(t))$ and $(u_2(x,t), s_2(t))$ be respectively lower and upper solutions of (2.1) corresponding respectively to the data (h_1, u_{01}, b_1) and (h_2, u_{02}, b_2).

If $b_1 < b_2$, $h_1 \leqslant h_2$ and $u_{01} \leqslant u_{02}$, then $s_1(t) < s_2(t)$ for $t \geqslant 0$ and $u_1(x,t) \leqslant u_2(x,t)$ for $x \geqslant 0$ and $t \geqslant 0$.

In particular, $u_1(x,t) < u_2(x,t)$ for $0 < x \leqslant s_1(t)$ and $t > 0$.

Before proving Theorem 3.2, we first show the following result.

Lemma 3.3. Any upper solution $(\bar{u}, \bar{s})$ of Problem (2.1) is such that $\bar{u} > 0$ in Q_T.

Proof. We first perform the change of function $\bar{u}(x,t) = \bar{v}(x,t)e^{\lambda t}$ where λ is a strictly positive constant. The function $\bar{v}$, as is easily seen, satisfies the inequality

$$(\bar{v}_t - \bar{v}_{xx} + \lambda \bar{v})e^{\lambda t} \geqslant 0 \ \text{ in } Q_T, \text{ for all } \lambda > 0,$$

so that

$$\bar{v}_t - (\bar{v}_{xx} - \lambda \bar{v}) \geqslant 0 \ \text{ in } Q_T, \text{ for all } \lambda > 0.$$

Now, we prove that $\bar{v} \geqslant 0$ in $\overline{Q_T}$. Indeed, it follows from the weak maximum principle (Lemma 1 of [F3, p.34]) that $\bar{v}$ cannot have a negative minimum in Q_T. Then, $\bar{v}$ attains its minimum on the boundary $\Gamma := \{(0,t), 0 \leqslant t \leqslant T\} \cup \{(x,0), 0 < x < b_0\} \cup \{(s(t),t), 0 \leqslant t \leqslant T\}$; since $\bar{v} \geqslant 0$ on Γ, it follows that $\bar{v} \geqslant 0$ in $\overline{Q_T}$ which implies that $\bar{u} \geqslant 0$ in $\overline{Q_T}$.

Next, we apply the strong maximum principle (Theorem 1 of [F3, p.34]) which states that if $\bar{v}$ attains its negative minimum at an interior point $(\bar{x}, \bar{t}) \in Q_T$, then $\bar{v}$ is constant in $Q_{\bar{t}}$. However, since $\bar{v}(0,t) \geqslant he^{-\lambda t} > 0$ for $t \in (0,T]$, we have reached a contradiction, so that we conclude that $\bar{v} > 0$ in Q_T. Then, we conclude that $\bar{u} > 0$ in Q_T. □

Proof of Theorem 3.2. Suppose that

$$\text{there exists } t_0 > 0 \text{ such that } s_1(t) < s_2(t) \text{ for } 0 \leqslant t < t_0 \text{ and } s_1(t_0) = s_2(t_0). \tag{3.2}$$

Let $x_0 := s_1(t_0)$. Since $s_1(t) < s_2(t)$ for $0 \leqslant t < t_0$, we see that

$$s_1'(t_0) \geqslant s_2'(t_0). \tag{3.3}$$

Let $D := \{(x,t) |\ 0 < t \leqslant t_0,\ 0 < x < s_1(t)\}$ and $\Gamma := \{(0,t) |\ 0 \leqslant t \leqslant t_0\} \cup \{(x,0) |\ 0 < x < b_1\} \cup \{(s_1(t),t) |\ 0 \leqslant t \leqslant t_0\}$. We introduce $w(x,t) := u_2(x,t) - u_1(x,t)$. We shall prove that $w > 0$ in

D. Indeed, $w_t - w_{xx} \geqslant 0$ in D, it follows from the weak maximum principle that $w \geqslant 0$ in $\bar{D}$. Then, we remark that $w(s_1(t), t) = u_2(s_1(t), t)$ and according to Lemma 3.3 we have $u_2(s_1(t), t) > 0$, so that we deduce from the strong maximum principle that $w > 0$ in D.

Let $\xi > 0$, $a := \xi^{-2}$,

$$\varphi(x,t) := e^{-a(x-x_0+\xi)^2+a(t-t_0)} - e^{-a\xi^2} \tag{3.4}$$

and

$$\rho(x) := (x - x_0 + \xi)^2 - \xi^2 + t_0. \tag{3.5}$$

Let $\delta > 0$ be small (to be chosen later). We define

$$D(\delta) := \{(x,t)|\ x_0 - \delta < x < x_0,\ \rho(x) < t < t_0\}.$$

Next we show that there exist a small $\xi > 0$ and a small $\delta_1 > 0$ such that $D(\delta_1) \subset D$, indeed since $0 < s_1'(t_0) < \infty$ and there exists a small $\xi > 0$ such that

$$s_1'(t_0) < \left.\frac{d\rho^{-1}(t)}{dt}\right|_{t=t_0} = \frac{1}{\rho'(x_0)} = \frac{1}{2\xi}.$$

It follows that if $\xi < \dfrac{1}{2s_1'(t_0)}$ then $D(\delta_1) \subset D$. Indeed, suppose that

$$\rho(x) := (x - x_0 + \xi)^2 - \xi^2 + t_0 = t.$$

Then

$$\frac{d\rho^{-1}(t)}{dt} = \frac{1}{\rho'(\rho^{-1}(t))} = \frac{1}{\rho'(x)},$$

where $\rho^{-1}(t)$ is the inverse function of $\rho(x)$ near $x = x_0$ and $\rho'(x) = 2(x - x_0 + \xi)$ which implies that $\rho'(x_0) = 2\xi$. By direct calculation, we shall prove that

$$\varphi_t(x_0,t_0) - \varphi_{xx}(x_0,t_0) = -e^{-1}\xi^{-2} < 0. \tag{3.6}$$

Indeed, from (3.4) we deduce that

$$\varphi_t(x,t) = ae^{-a(x-x_0+\xi)^2+a(t-t_0)} = a\varphi(x,t) + ae^{-a\xi^2}.$$

We remark that since $\varphi(x_0,t_0) = 0$, it follows that $\varphi_t(x_0,t_0) = ae^{-a\xi^2} = \xi^{-2}e^{-1}$. Next, we compute the space derivatives of φ :

$$\varphi_x(x,t) = -2a(x - x_0 + \xi)e^{-a(x-x_0+\xi)^2+a(t-t_0)},$$

$$\varphi_{xx}(x,t) = -2ae^{-a(x-x_0+\xi)^2+a(t-t_0)} + 4a^2(x - x_0 + \xi)^2e^{-a(x-x_0+\xi)^2+a(t-t_0)}.$$

Thus, $\varphi_{xx}(x_0,t_0) = -2ae^{-a\xi^2} + 4a^2\xi^2e^{-a\xi^2}$ and since $a = \xi^{-2}$, we have,

$$\varphi_{xx}(x_0,t_0) = -2\xi^{-2}e^{-1} + 4\xi^{-2}e^{-1} = 2\xi^{-2}e^{-1},$$

which implies (3.6).

Since φ is smooth, and since φ satisfies (3.6), there exists a neighborhood U of (x_0,t_0) such that $\varphi_t - \varphi_{xx} < 0$ in U. We choose $\delta_2 \in (0,\delta_1)$ such that $D(\delta_2) \subset U$.

We define $z(x,t) := w(x,t) - \varepsilon\varphi(x,t)$, where $\varepsilon > 0$ will be chosen later. Then,

$$z_t - z_{xx} \geqslant 0 \text{ on } D(\delta_2). \tag{3.7}$$

Indeed, since $w_t - w_{xx} \geqslant 0$ in D, $\varphi_t - \varphi_{xx} < 0$ in U and $D(\delta_2) \subset U$, we have

$$z_t - z_{xx} = w_t - \varepsilon\varphi_t - w_{xx} + \varepsilon\varphi_{xx} = w_t - w_{xx} + \varepsilon(\varphi_{xx} - \varphi_t) \geqslant 0 + \varepsilon(\varphi_{xx} - \varphi_t) > 0 \text{ in } D(\delta_2).$$

Let

$$\gamma_0 := \{(x,t)|\ x_0 - \delta_2 \leqslant x \leqslant x_0,\ t = \rho(x)\}$$

and

$$\gamma_1 := \{(x,t)|\ x = x_0 - \delta_2,\ \rho(x_0 - \delta_2) \leqslant t < t_0\}.$$

In what follows, we use the notation $\partial(D(\delta_2)) := \gamma_0 \cup \gamma_1$ to denote the parabolic boundary of $D(\delta_2)$. Next, we show that $\varphi = 0$ on γ_0. Indeed $t = \rho(x)$ on γ_0 , we have that

$$\varphi(x, \rho(x)) = e^{-a(x-x_0+\xi)^2 + a(x-x_0+\xi)^2} e^{-a\xi^2} - e^{-a\xi^2} = 0$$

and thus, $\varphi = 0$ on γ_0. Since $w \geq 0$ in $\bar{D}$ and $\gamma_0 \subset \bar{D}$, we deduce from the definition of z that $z = w \geqslant 0$ on γ_0.

Since $w > 0$ on γ_1, there exists a small $\varepsilon > 0$ such that $z \geqslant 0$ on γ_1. Indeed, $w > 0$ on D, so, there exists $\mu > 0$ such that $w \geqslant \mu$ in γ_1. Moreover, from (3.4) we deduce that

$$\varphi(x,t) \leqslant e^{-a(x-x_0+\xi)^2 + a(t-t_0)}$$

so that $\varphi(x,t) \leqslant 1$ and $\varepsilon\ \varphi(x,t) \leqslant \varepsilon$. Therefore, if

$$\varepsilon \leqslant \frac{\mu}{2},$$

we have

$$w \geqslant \mu \geqslant \varepsilon\varphi(x,t)$$

which implies that $z \geqslant 0$ on γ_1. Using the fact that $\partial(D(\delta_2)) = \gamma_0 \cup \gamma_1$ and $z \geqslant 0$ on $\partial(D(\delta_2))$, we deduce from the weak maximum principle together with (3.7) that $z \geqslant 0$ in $\overline{D(\delta_2)}$ and hence $w(x,t_0) \geqslant \varepsilon\varphi(x,t_0)$ for $x_0 - \delta_2 \leqslant x \leqslant x_0$. Thus,

$$z(x,t_0) \geqslant 0 \text{ for all } x \in [x_0 - \delta_2, x_0]. \tag{3.8}$$

Moreover, since (x_0, t_0) both belongs to s_1 and s_2, it follows that

$$z(x_0,t_0) = w(x_0,t_0) = u_2(x_0,t_0) - u_1(x_0,t_0) = 0. \tag{3.9}$$

We deduce from (3.8) and (3.9) that $z_x(x_0,t_0) \leqslant 0$, or else,

$$w_x(x_0,t_0) \leqslant \varepsilon\varphi_x(x_0,t_0) = -2\varepsilon e^{-1}\xi^{-1} < 0,$$

and hence $u_{1x}(x_0,t_0) > u_{2x}(x_0,t_0)$. Because of (3.1), we see that $s_1'(t_0) < s_2'(t_0)$. This contradicts (3.3). Since we have obtained a contradiction, (3.2) cannot occur. We see that $s_1(t) < s_2(t)$ for all $t \geqslant 0$. By the weak maximum principle we see that $u_1(x,t) \leqslant u_2(x,t)$ for $x \geqslant 0$ and $t \geqslant 0$. It follows from the strong maximum principle that $u_1(x,t) < u_2(x,t)$ for $0 < x < s_1(t)$ and $t > 0$. □

Next we present an extension of Theorem 3.2 for the case that $b_1 \leqslant b_2$.

Corollary 3.4 (Extension of the comparison principle)**.** Let $(u_1(x,t), s_1(t))$ and $(u_2(x,t), s_2(t))$ be respectively lower and upper solutions of (2.1) corresponding respectively to the data (h_1, u_{01}, b_1) and (h_2, u_{02}, b_2) such that u_{01} or u_{02} is a nonincreasing function.

If $b_1 \leqslant b_2$, $h_1 \leqslant h_2$ and $u_{01} \leqslant u_{02}$, then $s_1(t) \leqslant s_2(t)$ for $t \geqslant 0$ and $u_1(x,t) \leqslant u_2(x,t)$ for $x \geqslant 0$ and $t \geqslant 0$.

Proof. The case where $b_1 < b_2$ has already been studied. It only remains to study the case "$b_1 = b_2$". We start to suppose that u_{01} is nonincreasing. The case where u_{02} is nonincreasing will be considered after.

We will construct a lower solution $(u_\varepsilon, s_\varepsilon)$, $0 < \varepsilon < 1$, of Problem (2.1) corresponding to the data $(h_1, b_{0\varepsilon}, u_{0\varepsilon})$ such that $(b_{0\varepsilon}, u_{0\varepsilon})$ satisfies

$$\begin{cases} b_{0\varepsilon} < b_2 \quad \text{and} \quad b_{0\varepsilon} \to b_1 = b_2 \ \text{ as } \ \varepsilon \to 1, \\ u_{0\varepsilon} \leqslant u_{02}, \end{cases} \tag{3.10}$$

and

$$\begin{cases} s_\varepsilon(t) \xrightarrow[\varepsilon\to 1]{} s_1(t), & t \geqslant 0, \\ u_\varepsilon(x,t) \xrightarrow[\varepsilon\to 1]{} u_1(x,t), & x \geqslant 0, t \geqslant 0. \end{cases} \tag{3.11}$$

Then, it follows from (3.10) and Theorem 3.2 that

$$\begin{cases} s_\varepsilon(t) < s_2(t), & t \geqslant 0, \\ u_\varepsilon(x,t) \leqslant u_2(x,t), & x \geqslant 0, t \geqslant 0. \end{cases} \tag{3.12}$$

Letting $\varepsilon \to 1$, we obtain

$$\begin{cases} s_1(t) \leqslant s_2(t), & t \geqslant 0, \\ u_1(x,t) \leqslant u_2(x,t), & x \geqslant 0, t \geqslant 0. \end{cases} \tag{3.13}$$

Next, we complete the proof by the construction of a lower solution $(u_\varepsilon, s_\varepsilon)$ which satisfies (3.11) with data $(h_1, u_{0\varepsilon}, b_{0\varepsilon})$ such that $(u_{0\varepsilon}, b_{0\varepsilon})$ satisfies (3.10).

Construction of the lower solution $(u_\varepsilon, s_\varepsilon)$. We choose

$$\begin{cases} s_\varepsilon(t) = \varepsilon \,.\, s_1\big(\dfrac{t}{\varepsilon^2}\big), & t \geqslant 0, \\ u_\varepsilon(x,t) = u_1\left(\dfrac{x}{\varepsilon}, \dfrac{t}{\varepsilon^2}\right), & x \geqslant 0, t \geqslant 0. \end{cases} \tag{3.14}$$

We first check that $(u_\varepsilon, s_\varepsilon)$ corresponding to the data $(h_1, u_{0\varepsilon}, b_{0\varepsilon})$ is a lower solution of (2.1). Indeed, since u_1 is a lower solution of (2.1), it follows that

$$\mathcal{L}(u_\varepsilon) = \frac{1}{\varepsilon^2}\left(u_{1,t}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) - u_{1,xx}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)\right) \leqslant 0, \tag{3.15}$$

$$u_\varepsilon(0,t) = u_1\big(0, \frac{t}{\varepsilon^2}\big) = h_1 \leqslant h, \tag{3.16}$$

$$u_\varepsilon\big(s_\varepsilon(t), t\big) = u_1\left(s_1\left(\frac{t}{\varepsilon^2}\right), \frac{t}{\varepsilon^2}\right) = 0, \tag{3.17}$$

$$\frac{ds_\varepsilon(t)}{dt} = \frac{1}{\varepsilon}\frac{d}{dt}s_1\left(\frac{t}{\varepsilon^2}\right) = \frac{-1}{\varepsilon}\, u_{1,x}\left(s_1\left(\frac{t}{\varepsilon^2}\right), \frac{t}{\varepsilon^2}\right) = -u_{\varepsilon,x}(s_\varepsilon(t), t). \tag{3.18}$$

Next, we choose data $(h_1, u_{0\varepsilon}, b_{0\varepsilon})$ such that $(u_{0\varepsilon}, b_{0\varepsilon})$ satisfies (3.10). We set

$$b_{0\varepsilon} := \varepsilon b_1. \tag{3.19}$$

Then, it follows from (3.19) and $0 < \varepsilon < 1$ that

$$s_\varepsilon(0) = \varepsilon s_1(0) = \varepsilon b_1 =: b_{0\varepsilon} < b_1. \tag{3.20}$$

Finally, we should check that $u_\varepsilon(x, 0) := u_{0\varepsilon}$ satisfies the second condition of (3.10). Indeed, we have

$$u_\varepsilon(x,0) = u_1\big(\frac{x}{\varepsilon}, 0\big) = u_{01}\big(\frac{x}{\varepsilon}\big). \tag{3.21}$$

Since u_{01} is a nonincreasing function and $0 < \varepsilon < 1$, it follows that

$$u_{01}\big(\frac{x}{\varepsilon}\big) \leqslant u_{01}(x) \ \text{ for } x \geqslant 0. \tag{3.22}$$

We deduce from (3.22) that

$$u_\varepsilon(x,0) := u_{0\varepsilon}(x) = u_{01}\big(\frac{x}{\varepsilon}\big) \leqslant u_{01}(x) \leqslant u_{02}(x) \ \text{ for } x \geqslant 0. \tag{3.23}$$

Therefore,$(u_\varepsilon, s_\varepsilon)$ satisfies (3.15)-(3.18) and corresponds to data $(h_1, u_{0\varepsilon}, b_{0\varepsilon})$ such that $(u_{0\varepsilon}, b_{0\varepsilon})$ satisfies (3.10). Thus, it is a lower solution of (2.1).

Now, we consider the case where the function u_{02} is nonincreasing. We can proceed exactly as before by considering the upper solution $(u_\varepsilon, s_\varepsilon)$ of Problem (2.1) with $\varepsilon > 1$, given by

$$\begin{cases} s_\varepsilon(t) = \varepsilon \ . \ s_2\big(\frac{t}{\varepsilon^2}\big), & t \geqslant 0, \\ u_\varepsilon(x,t) = u_2\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right), & x \geqslant 0, t \geqslant 0. \end{cases} \tag{3.24}$$

The corresponding initial datum $b_{0\varepsilon} = s_\varepsilon(0)$ and $u_{0\varepsilon} = u_\varepsilon(x, 0)$ verify

$$\begin{cases} b_1 < b_{0\varepsilon} \quad \text{and} \quad b_{0\varepsilon} \to b_2 = b_1 \ \text{ as } \ \varepsilon \to 1, \\ u_{01} \leqslant u_{0\varepsilon}, \end{cases} \tag{3.25}$$

and

$$\begin{cases} s_\varepsilon(t) \xrightarrow[\varepsilon\to1]{} s_2(t), & t \geqslant 0, \\ u_\varepsilon(x,t) \xrightarrow[\varepsilon\to1]{} u_2(x,t), & x \geqslant 0, t \geqslant 0. \end{cases} \tag{3.26}$$

Then, the result follows from the use of Theorem 3.2 with (3.25) and letting $\varepsilon \to 1$. □

4 Self-similar solution

We now look for a self-similar solution of the problem

$$\begin{cases} u_t = u_{xx}, & t > 0, \quad 0 < x < s(t), \\ u(0,t) = h, & t > 0, \\ u(s(t),t) = 0, & t > 0, \\ \dfrac{ds(t)}{dt} = -u_x(s(t),t), & t > 0, \end{cases} \tag{4.1}$$

in the form

$$\begin{cases} u(x,t) = U\left(\dfrac{x}{\sqrt{t+1}}\right), \\ s(t) = a\sqrt{t+1}, \end{cases} \tag{4.2}$$

for some positive constant a still to be determined. We set

$$\eta := \frac{x}{\sqrt{t+1}}. \tag{4.3}$$

and deduce that

$$\begin{cases} U_{\eta\eta} + \dfrac{\eta}{2} U_\eta = 0, & 0 < \eta < a, \\ U(0) = h, & U(a) = 0. \end{cases} \tag{4.4}$$

The unique solution of (4.4) is given by

$$U(\eta) = h\left[1 - \frac{\int_0^\eta e^{-\frac{s^2}{4}}\,ds}{\int_0^a e^{-\frac{s^2}{4}}\,ds}\right] \quad \text{for all } \eta \in (0,a). \tag{4.5}$$

It remains to determine the constant a. We write that

$$s'(t) = \frac{a}{2\sqrt{t+1}} = -u_x(s(t),t) = -\frac{U_\eta\left(\dfrac{s(t)}{\sqrt{t+1}}\right)}{\sqrt{t+1}}, \tag{4.6}$$

which implies that

$$\frac{a}{2} = -U_\eta(a), \tag{4.7}$$

so that a is characterized as the unique solution of the equation

$$h = \frac{a}{2} e^{\frac{a^2}{4}} \int_0^a e^{-\frac{s^2}{4}}\,ds. \tag{4.8}$$

We remark that the function $a = a(h)$ is strictly increasing, which in turn implies that the functional $h \to U$ is strictly increasing.

We conclude that the self-similar solution of Problem (4.1) coincides with the unique solution (U,a) of Problem (1.12).

5 New coordinates, upper and lower solutions

We set

$$\begin{cases} V(\eta,t) = u(x,t), \\ a(t) = \dfrac{s(t)}{\sqrt{t+1}}, \end{cases} \tag{5.1}$$

and obtain the problem

$$\begin{cases} (t+1)V_t = V_{\eta\eta} + \dfrac{\eta}{2} V_\eta, & t > 0, \quad 0 < \eta < a(t), \\ V(0,t) = h, \quad V(a(t),t) = 0, & t > 0, \\ (t+1)\dfrac{da(t)}{dt} + \dfrac{a(t)}{2} = -V_\eta(a(t),t), & t > 0. \end{cases} \tag{5.2}$$

Finally we set

$$\tau = \ln(t+1).$$

The equations in the system (5.2) read as

$$\begin{cases} W_\tau = W_{\eta\eta} + \dfrac{\eta}{2} W_\eta, & \tau > 0, \quad 0 < \eta < b(\tau), \\ W(0,\tau) = h, \quad W(b(\tau),\tau) = 0, & \tau > 0, \\ \dfrac{db(\tau)}{d\tau} + \dfrac{b(\tau)}{2} = -W_\eta(b(\tau),\tau), & \tau > 0, \end{cases} \tag{5.3}$$

where we have set

$$V(\eta,t) = W(\eta,\tau), \; a(t) = b(\tau).$$

Next, we write the full time evolution problem corresponding to the system (5.3). It is given by

$$\begin{cases} W_\tau = W_{\eta\eta} + \dfrac{\eta}{2} W_\eta, & \tau > 0, \quad 0 < \eta < b(\tau), \\ W(0,\tau) = h, & \tau > 0, \\ W(b(\tau),\tau) = 0, & \tau > 0, \\ \dfrac{db(\tau)}{d\tau} + \dfrac{b(\tau)}{2} = -W_\eta(b(\tau),\tau), & \tau > 0, \\ b(0) = b_0, & \\ W(\eta,0) = u_0(\eta), & 0 \leqslant \eta \leqslant b_0. \end{cases} \tag{5.4}$$

Finally, we note that the stationary solution of Problem (5.4) coincides with the unique solution of Problem (1.12), or in other words, the self-similar solution of Problem (1.1).

Definition 5.1. We define the linear operator $\mathcal{L}(W) := W_\tau - W_{\eta\eta} - \dfrac{\eta}{2} W_\eta$.
$(\underline{W}, \underline{b})$ is a lower solution of Problem (5.4) if it satisfies:

$$\begin{cases} \mathcal{L}(\underline{W}) = \underline{W}_\tau - \underline{W}_{\eta\eta} - \dfrac{\eta}{2} \underline{W}_\eta \leqslant 0, & \tau > 0, \quad 0 < \eta < \underline{b}(\tau), \\ \underline{W}(0,\tau) \leqslant h, \quad \underline{W}(\underline{b}(\tau),\tau) = 0, & \tau > 0, \\ \dfrac{d\underline{b}(\tau)}{d\tau} + \dfrac{\underline{b}(\tau)}{2} = -\underline{W}_\eta(\underline{b}(\tau),\tau), & \tau > 0, \\ \underline{b}(0) \leqslant b_0, & \\ \underline{W}(\eta,0) \leqslant u_0(\eta), & 0 \leqslant \eta \leqslant \underline{b}(0). \end{cases} \tag{5.5}$$

$(\bar{W}, \bar{b})$ is an upper solution of the Problem (5.4) if it satisfies Problem (5.5) with all $\leqslant$ replaced with $\geqslant$.

Finally, we deduce from Corollary 3.4 that the following comparison principle holds.

Theorem 5.2. Let $\big(W_1(\eta,\tau), b_1(\tau)\big)$ and $\big(W_2(\eta,\tau), b_2(\tau)\big)$ be respectively lower and upper solutions of (5.4) corresponding respectively to the data (h_1, u_{01}, b_{01}) and (h_2, u_{02}, b_{02}) such that u_{01} or u_{02} is a nonincreasing function.

If $b_{01} \leqslant b_{02}$, $h_1 \leqslant h_2$ and $u_{01} \leqslant u_{02}$, then $b_1(\tau) \leqslant b_2(\tau)$ for $\tau \geqslant 0$ and $W_1(\eta,\tau) \leqslant W_2(\eta,\tau)$ for $\eta \geqslant 0$ and $\tau \geqslant 0$.

Throughout this paper, we will also make use of the explicit notation $W\big(\eta,\tau,(u_0,b_0)\big)$ and $b\big(\tau,(u_0,b_0)\big)$ for the solution pair associated with the initial data (u_0, b_0).

5.1 Construction of upper and lower solutions

In this section, we construct ordered upper and lower solutions for Problem (5.4). For $\lambda \geqslant 0$, we consider (W_λ, b_λ) the unique solution of the problem

$$\begin{cases} W_{\eta\eta} + \dfrac{\lambda\eta}{2} W_\eta = 0, & 0 < \eta < b, \\ W(0) = h, \quad W(b) = 0, \\ \dfrac{b}{2} = -W_\eta(b), \end{cases} \tag{5.6}$$

which is given by

$$W_\lambda(\eta) = h\left[1 - \frac{\int_0^\eta e^{-\frac{\lambda s^2}{4}} ds}{\int_0^{b_\lambda} e^{-\frac{\lambda s^2}{4}} ds}\right] \quad \text{for all } \eta \in (0, b_\lambda) \tag{5.7}$$

and b_λ is the unique solution of the equation

$$h = \frac{b_\lambda}{2} e^{\frac{\lambda b_\lambda^2}{4}} \int_0^{b_\lambda} e^{-\frac{\lambda s^2}{4}} ds. \tag{5.8}$$

We can easily show the following properties for (W_λ, b_λ).

Lemma 5.3. We have that

$$0 \leqslant W_\lambda(\eta) \leqslant h \quad \text{for all } \lambda \geqslant 0 \text{ and } 0 \leqslant \eta \leqslant b_\lambda, \tag{5.9}$$

$$W_{\lambda,\eta}(\eta) \leqslant 0 \quad \text{for all } \lambda \geqslant 0 \text{ and } 0 \leqslant \eta \leqslant b_\lambda \tag{5.10}$$

and

$$W_{\lambda,\eta\eta}(\eta) \geqslant 0 \quad \text{for all } \lambda \geqslant 0 \text{ and } 0 \leqslant \eta \leqslant b_\lambda. \tag{5.11}$$

In particular,

$$W_\lambda \text{ is } \begin{cases} \text{a linear function} & \text{if } \lambda = 0, \\ \text{a convex function} & \text{if } \lambda > 0, \end{cases} \tag{5.12}$$

and

$$b_\lambda = \begin{cases} \sqrt{2h} & \text{if } \lambda = 0, \\ \text{satisfies the equation (5.8)} & \text{if } \lambda > 0. \end{cases} \tag{5.13}$$

Lower solution. We suppose that

$$\lambda \geqslant 1, \tag{5.14}$$

then (W_λ, b_λ) is a lower solution for Problem (5.4). Indeed, we easily check that W_λ satisfies the following property

$$-W_{\lambda,\eta\eta} - \frac{\eta}{2} W_{\lambda,\eta} \leqslant 0 \quad \text{if and only if} \quad \lambda \geqslant 1. \tag{5.15}$$

We define $(\underline{W}_\lambda, \underline{b}_\lambda)$ by

$$\underline{b}_\lambda = b_\lambda \quad \text{and} \quad \underline{W}_\lambda(\eta) := \begin{cases} W_\lambda(\eta) & \text{if } 0 \leqslant \eta \leqslant \underline{b}_\lambda, \\ 0 & \text{if } \eta > \underline{b}_\lambda. \end{cases} \tag{5.16}$$

The pair $(\underline{W}_\lambda, \underline{b}_\lambda)$ is a lower solution for Problem (5.4).

We assume the following condition on the initial data (u_0, b_0):

$$\underline{\mathcal{W}}_\lambda(\eta) \leqslant u_0(\eta) \text{ for all } 0 \leqslant \eta \leqslant b_0, \quad \underline{b}_\lambda \leqslant b_0. \tag{5.17}$$

According to (5.10), $\underline{\mathcal{W}}_\lambda$ is a nonincreasing function and then, in view of the comparison principle Theorem 5.2, it follows that

$$\underline{b}_\lambda \leqslant b\big(\tau, (u_0, b_0)\big) \text{ and } \underline{\mathcal{W}}_\lambda(\eta) \leqslant W\big(\eta, \tau, (u_0, b_0)\big) \text{ for all } \tau \geqslant 0, \eta \geqslant 0. \tag{5.18}$$

<u>**Upper solution.**</u> Now, we suppose that

$$0 \leqslant \lambda \leqslant 1. \tag{5.19}$$

We define $(\bar{\mathcal{W}}_\lambda, \bar{b}_\lambda)$ by

$$\bar{b}_\lambda = b_\lambda \quad \text{and} \quad \bar{\mathcal{W}}_\lambda(\eta) := \begin{cases} W_\lambda(\eta) & \text{if } 0 \leqslant \eta \leqslant \bar{b}_\lambda, \\ 0 & \text{if } \eta > \bar{b}_\lambda. \end{cases} \tag{5.20}$$

In view of (5.15), the pair $(\bar{\mathcal{W}}_\lambda, \bar{b}_\lambda)$ is an upper solution for Problem (5.4). We now suppose that $\lambda = 0$ and define the corresponding upper solution by $(\bar{\mathcal{W}}, \bar{b})$

$$\bar{b} = \sqrt{2h} \quad \text{and} \quad \bar{\mathcal{W}}(\eta) := \begin{cases} W_0(\eta) & \text{if } 0 \leqslant \eta \leqslant \bar{b}, \\ 0 & \text{if } \eta > \bar{b}, \end{cases} \tag{5.21}$$

where $W_0(\eta) = h\big(1 - \dfrac{\eta}{\sqrt{2h}}\big)$ for all $0 < \eta < \bar{b}$.

We assume the following condition on the initial data (u_0, b_0):

$$u_0(\eta) \leqslant \bar{\mathcal{W}}(\eta) \text{ for all } 0 \leqslant \eta \leqslant \bar{b}, \quad b_0 \leqslant \bar{b}. \tag{5.22}$$

According to (5.10), $\bar{\mathcal{W}}$ is a nonincreasing function and then, in view of the comparison principle Theorem 5.2, it follows that

$$b\big(\tau, (u_0, b_0)\big) \leqslant \bar{b} \text{ and } W\big(\eta, \tau, (u_0, b_0)\big) \leqslant \bar{\mathcal{W}}(\eta) \text{ for all } \tau \geqslant 0, \eta \geqslant 0. \tag{5.23}$$

Next, we prove the monotonicity in time of the solution pair (W, b) of the time evolution Problem (5.4) with the two initial conditions $(\bar{\mathcal{W}}, \bar{b})$ and $(\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)$.

Lemma 5.4. Suppose that the initial data (u_0, b_0) satisfies (5.17) and (5.22). Let $(\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)$ and $\big(\bar{\mathcal{W}}, \bar{b}\big)$ be defined by (5.16) and (5.21).

(i) The functions $W\big(\eta, \tau, (\bar{\mathcal{W}}, \bar{b})\big)$ and $b\big(\tau, (\bar{\mathcal{W}}, \bar{b})\big)$ are nonincreasing in time. Furthermore, there exist a positive constant $\bar{b}_\infty$ and a function $\phi \in L^\infty(0, \bar{b}_\infty)$ such that

$$\lim_{\tau \to +\infty} W\big(\eta, \tau, (\bar{\mathcal{W}}, \bar{b})\big) = \phi(\eta) \quad \text{for all } \eta \in (0, \bar{b}_\infty), \tag{5.24}$$

$$\lim_{\tau \to +\infty} b\big(\tau, (\bar{\mathcal{W}}, \bar{b})\big) = \bar{b}_\infty. \tag{5.25}$$

(ii) The function $W\big(\eta, \tau, (\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)\big)$ and $b\big(\tau, (\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)\big)$ are nondeacreasing in time. Furthermore, there exist a positive constant $\underline{b}_\infty$ and a function $\psi \in L^\infty(0, \underline{b}_\infty)$ such that

$$\lim_{\tau \to +\infty} W(\eta, \tau, (\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)) = \psi(\eta) \quad \text{for all } \eta \in (0, \underline{b}_\infty), \tag{5.26}$$

$$\lim_{\tau \to +\infty} b\big(\tau, (\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)\big) = \underline{b}_\infty. \tag{5.27}$$

Proof. Applying repeatedly Theorem 5.2, one can show that $W\big(\eta,\tau,(\bar{\mathcal{W}},\bar{b})\big)$ and $b\big(\tau,(\bar{\mathcal{W}},\bar{b})\big)$ are nonincreasing in time and that $W\big(\eta,\tau,(\underline{\mathcal{W}}_\lambda,\underline{b}_\lambda)\big)$ and $b\big(\tau,(\underline{\mathcal{W}}_\lambda,\underline{b}_\lambda)\big)$ are nondeacreasing in time. Indeed, from (5.23) we have that

$$b\big(\tau,(u_0,b_0)\big)\leqslant\bar{b}\ \text{ and } W\big(\eta,\tau,(u_0,b_0)\big)\leqslant\bar{\mathcal{W}}(\eta) \text{ for all } \tau\geqslant 0 \text{ and } \eta\geqslant 0.$$

In particular, with $u_0=\bar{\mathcal{W}}$ and $b_0=\bar{b}$, we get

$$b\big(\tau,(\bar{\mathcal{W}},\bar{b})\big)\leqslant\bar{b} \text{ and } W\big(\eta,\tau,(\bar{\mathcal{W}},\bar{b})\big)\leqslant\bar{\mathcal{W}}(\eta) \quad \text{for all } \tau\geqslant 0 \text{ and } \eta\geqslant 0. \tag{5.28}$$

From (5.9), we have that

$$0\leqslant\bar{\mathcal{W}}(\eta)\leqslant h.$$

Then, it follows from Proposition 2.3 that

$$0\leqslant W\big(\eta,\tau,(\bar{\mathcal{W}},\bar{b})\big)\leqslant h \quad \text{for all } \tau\geqslant 0 \text{ and } \eta\geqslant 0. \tag{5.29}$$

Let $\sigma>0$ be fixed. Due to (5.10), $\bar{\mathcal{W}}$ is a nonincreasing function, then we apply Theorem 5.2 for (5.28) to obtain

$$b\big(\tau+\sigma,(\bar{\mathcal{W}},\bar{b})\big)\leqslant b\big(\sigma,(\bar{\mathcal{W}},\bar{b})\big) \text{ and } W\big(\eta,\tau+\sigma,(\bar{\mathcal{W}},\bar{b})\big)\leqslant W\big(\eta,\sigma,(\bar{\mathcal{W}},\bar{b})\big) \text{ for all } \ \tau\geqslant 0 \text{ and } \eta\geqslant 0.$$

Thus for each $\eta, W\big(\eta,\tau,(\bar{\mathcal{W}},\bar{b})\big)$ is nonincreasing in τ and from (5.29), it is bounded from below by zero. Therefore it has a limit ϕ as $\tau\to\infty$.

Also $b\big(\tau,(\bar{\mathcal{W}},\bar{b})\big)$ is nonincreasing in τ and from (5.18) we deduce that it is bounded from below by $\underline{b}_\lambda$. Therefore it has a limit $\bar{b}_\infty$ as $\tau\to\infty$.

The same reasoning can be applied to prove that $W\big(\eta,\tau,(\underline{\mathcal{W}}_\lambda,\underline{b}_\lambda)\big)$ and $b\big(\tau,(\underline{\mathcal{W}}_\lambda,\underline{b}_\lambda)\big)$ are nondecreasing in time. Thus for each $\eta, W\big(\eta,\tau,(\underline{\mathcal{W}}_\lambda,\underline{b}_\lambda)\big)$ is nondecreasing in τ and it is bounded from above by the constant function h as follows from Proposition 2.3. Therefore it has a limit ψ as $\tau\to\infty$. Also, $b\big(\tau,(\underline{\mathcal{W}}_\lambda,\underline{b}_\lambda)\big)$ is nondecreasing in τ and bounded from above by $\bar{b}$ thanks to (5.23). Therefore it has a limit $\underline{b}_\infty$ as $\tau\to\infty$. □

Later we will show that ϕ and ψ coincide with the unique solution of Problem (1.12). To that purpose, we will derive in the Section 6 estimates for the free boundary Problem (5.4) in both moving and fixed domains.

5.2 Properties of a family of upper and lower solutions

In this subsection, we establish some further properties of upper and lower solutions through successive lemmas.

Lemma 5.5. The following properties hold for b_λ satisfying (5.8).

(i) b_λ is a decreasing function of λ.

(ii) $b_\lambda\to 0$ as $\lambda\to+\infty$.

Proof. We start to prove (i). We define $\mathcal{F}$ as the function given by

$$\mathcal{F}(\lambda,b_\lambda)=\frac{b_\lambda}{2}\int_0^{b_\lambda} e^{\frac{\lambda(b_\lambda^2-s^2)}{4}}ds-h \tag{5.30}$$

and consider the equation $\mathcal{F}(\lambda, b_\lambda) = 0$. We compute the differential of $\mathcal{F}$ through partial derivatives given by

$$d\mathcal{F} = \frac{\partial \mathcal{F}}{\partial \lambda} d\lambda + \frac{\partial \mathcal{F}}{\partial b_\lambda} db_\lambda. \tag{5.31}$$

From (5.30), it follows that

$$\frac{\partial \mathcal{F}}{\partial \lambda} = \frac{b_\lambda}{2} \int_0^{b_\lambda} \frac{(b_\lambda^2 - s^2)}{4} e^{\frac{\lambda(b_\lambda^2 - s^2)}{4}} ds > 0 \text{ for all } b_\lambda > 0, \tag{5.32}$$

and

$$\frac{\partial \mathcal{F}}{\partial b_\lambda} = \frac{1}{2} \int_0^{b_\lambda} e^{\frac{\lambda(b_\lambda^2 - s^2)}{4}} ds + \frac{b_\lambda}{2}\left(1 + \int_0^{b_\lambda} \frac{2b_\lambda \lambda}{4} e^{\frac{\lambda(b_\lambda^2 - s^2)}{4}} ds\right) > 0 \text{ for all } b_\lambda > 0. \tag{5.33}$$

Since $\mathcal{F}(\lambda, b_\lambda) = 0$, it follows from (5.31) that

$$\frac{\partial \mathcal{F}(\lambda, b_\lambda)}{\partial \lambda} d\lambda + \frac{\partial \mathcal{F}(\lambda, b_\lambda)}{\partial b_\lambda} db_\lambda = 0. \tag{5.34}$$

Thus, since $\dfrac{\partial \mathcal{F}}{\partial b_\lambda} \neq 0$, it follows from (5.32),(5.33) and (5.34) that

$$\frac{db_\lambda}{d\lambda} = -\frac{\dfrac{\partial \mathcal{F}(\lambda, b_\lambda)}{\partial \lambda}}{\dfrac{\partial \mathcal{F}(\lambda, b_\lambda)}{\partial b_\lambda}} < 0, \tag{5.35}$$

which completes the proof of (i).

Now, we turn to the proof of (ii). For $\lambda \geqslant 0$, we have $b_\lambda > 0$ and b_λ is a decreasing function of λ. Hence, there exists $\alpha \geqslant 0$ such that $b_\lambda \to \alpha$ as $\lambda \to +\infty$ and $b_\lambda \geqslant \alpha$ for all $\lambda \geqslant 0$. We shall prove that $\alpha = 0$. This fact mainly relies on the following inequality which will be proved later on. Let $a \geqslant 0$. For $\lambda \geqslant 0$ large enough, the following inequality holds:

$$\int_0^a e^{-\frac{\lambda s^2}{4}} ds \geqslant a(1 + \frac{\lambda}{4} a^2) e^{-\frac{\lambda a^2}{4}}. \tag{5.36}$$

Since $b_\lambda \geqslant \alpha$ for all $\lambda \geqslant 0$, we deduce from (5.8) that

$$h \geqslant \frac{\alpha}{2} e^{\frac{\lambda \alpha^2}{4}} \int_0^\alpha e^{-\frac{\lambda s^2}{4}} ds. \tag{5.37}$$

For λ large enough we infer from the estimate (5.36) that

$$h \geqslant \frac{\alpha^2}{2}(1 + \frac{\lambda}{4}\alpha^2). \tag{5.38}$$

Letting $\lambda \to +\infty$ in (5.38), we see that we necessarily have $\alpha = 0$. It remains to prove that the inequality (5.36) holds for λ large enough. We only have to consider the case where $a > 0$ since (5.36) is trivially true for $a = 0$. Let us introduce $f(x) = e^{-\frac{\lambda x^2}{4}}$. We have $f''(x) = \frac{\lambda}{2}(\frac{\lambda}{2}x^2 - 1)e^{-\frac{\lambda x^2}{4}}$. We choose $\lambda > 0$ large enough to have $0 < \sqrt{\frac{2}{\lambda}} < a$ and then f is convex in $[\sqrt{\frac{2}{\lambda}}, a]$. Therefore, for all $x \in [\sqrt{\frac{2}{\lambda}}, a]$ we have

$$f(x) \geqslant g(x) := f(a) + (x - a) f'(a) \tag{5.39}$$

that is

$$e^{-\frac{\lambda x^2}{4}} \geqslant \left(1+\frac{\lambda}{2}a(a-x)\right)e^{-\frac{\lambda a^2}{4}}, \quad \text{for all } x \in [\sqrt{\frac{2}{\lambda}}, a]. \tag{5.40}$$

Next we prove that (5.39) also holds for $x \in [0, \sqrt{\frac{2}{\lambda}}]$. Indeed, we have

$$\max_{x\in[0,\sqrt{\frac{2}{\lambda}}]} g(x) = g(0) = (1+\frac{\lambda}{2}a^2)e^{-\frac{\lambda a^2}{4}}$$

and

$$\min_{x\in[0,\sqrt{\frac{2}{\lambda}}]} f(x) = f(\sqrt{\frac{2}{\lambda}}) = e^{-\frac{1}{2}}.$$

Since $g(0) \to 0$ as $\lambda \to +\infty$, we get, for λ large enough

$$\max_{[0,\sqrt{\frac{2}{\lambda}}]} g = g(0) \leq \min_{[0,\sqrt{\frac{2}{\lambda}}]} f = e^{-\frac{1}{2}} \tag{5.41}$$

and then

$$g(x) \leqslant f(x), \text{ for all } x \in [0, \sqrt{\frac{2}{\lambda}}] \tag{5.42}$$

Combining (5.39) with (5.42) leads to $f(x) \geqslant g(x)$ for all $x \in [0, a]$, that is

$$e^{-\frac{\lambda x^2}{4}} \geqslant \left(1+\frac{\lambda}{2}a(a-x)\right)e^{-\frac{\lambda a^2}{4}}, \quad \text{for all } x \in [0, a]. \tag{5.43}$$

Integrating (5.43) over $[0, a]$ leads to the desired inequality (5.36). □

Next, we prove the following result.

Lemma 5.6. Let λ_1 and λ_2 be such that $\lambda_1 < \lambda_2$, then it follows that

$$b_{\lambda_1} > b_{\lambda_2}, \tag{5.44}$$

and

$$W_{\lambda_1}(\eta) \geqslant W_{\lambda_2}(\eta) \text{ for all } 0 \leqslant \eta \leqslant b_{\lambda_2}. \tag{5.45}$$

Proof. From Lemma 5.5, since $\lambda_1 < \lambda_2$, it follows that $b_{\lambda_1} > b_{\lambda_2}$. Then, (5.44) holds. Next, we show (5.45). To do so, let the pair $(W_{\lambda_i}, b_{\lambda_i})_{i\in\{1,2\}}$ be the unique solution of the problem

$$\begin{cases} W_{\lambda_i,\eta\eta} + \dfrac{\lambda_i\eta}{2}W_{\lambda_i,\eta} = 0, & 0 < \eta < b_{\lambda_i} \\ W_{\lambda_i}(0) = h, \quad W_{\lambda_i}(b_{\lambda_i}) = 0. \end{cases} \tag{5.46}$$

Then, we recall that for $i \in \{1, 2\}$ the solution pair $(W_{\lambda_i}, b_{\lambda_i})$ is given by

$$W_{\lambda_i}(\eta) = h\left[1 - \frac{\int_0^\eta e^{-\frac{\lambda_i s^2}{4}}ds}{\int_0^{b_{\lambda_i}} e^{-\frac{\lambda_i s^2}{4}}ds}\right] \quad \text{for all } 0 \leqslant \eta \leqslant b_{\lambda_i}, \tag{5.47}$$

with also

$$W_{\lambda_i,\eta}(\eta) = \frac{-h\, e^{-\frac{\lambda_i \eta^2}{4}}}{\int_0^{b_{\lambda_i}} e^{-\frac{\lambda_i s^2}{4}}ds} \quad \text{for all } 0 \leqslant \eta \leqslant b_{\lambda_i}. \tag{5.48}$$

Next, we define the linear operator $\mathcal{L}(W) := W_{\eta\eta} + \dfrac{\lambda_1 \eta}{2} W_\eta$ for all $0 \leqslant \eta \leqslant b_{\lambda_2}$. We compute $\mathcal{L}(W_{\lambda_2} - W_{\lambda_1})$ to obtain

$$\mathcal{L}(W_{\lambda_2} - W_{\lambda_1}) = W_{\lambda_2,\eta\eta} + \frac{\lambda_1 \eta}{2} W_{\lambda_2,\eta} - W_{\lambda_1,\eta\eta} - \frac{\lambda_1 \eta}{2} W_{\lambda_1,\eta} \text{ for all } 0 \leqslant \eta \leqslant b_{\lambda_2}. \tag{5.49}$$

Then, from (5.46), we have that

$$W_{\lambda_2,\eta\eta}(\eta) = -\frac{\lambda_2 \eta}{2} W_{\lambda_2,\eta} \text{ for all } 0 \leqslant \eta \leqslant b_{\lambda_2}. \tag{5.50}$$

We substitute (5.50) in (5.49). Then, since $(W_{\lambda_1}, b_{\lambda_1})$ is a solution of problem (5.46), (5.49) becomes

$$\mathcal{L}(W_{\lambda_2} - W_{\lambda_1}) = \frac{(\lambda_1 - \lambda_2)\,\eta}{2} W_{\lambda_2,\eta}. \tag{5.51}$$

Since $\lambda_1 < \lambda_2$, by (5.48) and (5.51), we deduce that

$$\mathcal{L}(W_{\lambda_2} - W_{\lambda_1}) \geqslant 0 \text{ for all } 0 \leqslant \eta \leqslant b_{\lambda_2}. \tag{5.52}$$

Then, from (5.46), since

$$\big(W_{\lambda_2} - W_{\lambda_1}\big)(0) = h - h = 0$$

and

$$\big(W_{\lambda_2} - W_{\lambda_1}\big)(b_{\lambda_2}) = 0 - W_{\lambda_1}(b_{\lambda_2}) < 0,$$

we deduce from the one-dimensional maximum principle (Theorem 1 of [PW, p.2]) that the function $W_{\lambda_2} - W_{\lambda_1}$ attains its maximum on the boundary. This implies that

$$W_{\lambda_2}(\eta) - W_{\lambda_1}(\eta) \leqslant 0 \text{ for all } 0 \leqslant \eta \leqslant b_{\lambda_2},$$

which completes the proof of Lemma 5.6. □

The next result ensures that the assumption made in (5.17) on the initial datum is fulfilled for λ large enough.

Lemma 5.7. Let $u_0 \in X^h(b_0) \cap \mathbb{W}^{1,\infty}\big(0, b_0\big)$ and $(\underline{W}_\lambda, \underline{b}_\lambda)$ defined by (5.16). There exists $\lambda \geqslant 1$ large enough such that $\underline{W}_\lambda \leqslant u_0$ in $[0, b_0]$ and $\underline{b}_\lambda \leqslant b_0$.

Proof. According to (5.12), W_λ is a convex function. Thus, we have

$$W_\lambda(\eta) \leqslant \frac{h}{b_\lambda}(b_\lambda - \eta) \quad \text{for all } 0 \leqslant \eta \leqslant b_\lambda. \tag{5.53}$$

From the identity $u_0(\eta) = h + \displaystyle\int_0^\eta \frac{du_0}{d\eta}(s)ds$ for $0 \leqslant \eta \leqslant b_0$, we deduce that

$$u_0(\eta) \geqslant h - M\eta \quad \text{for all } 0 \leqslant \eta \leqslant b_0 \tag{5.54}$$

where $M = \|\dfrac{du_0}{d\eta}\|_{L^\infty(0,b_0)}$. From Lemma 5.5 (ii), $b_\lambda \to 0$ as $\lambda \to +\infty$. Then we can choose $\lambda \geqslant 1$ large enough so that

$$b_\lambda \leqslant \min(\frac{h}{M}, b_0). \tag{5.55}$$

Estimate (5.54) then becomes

$$u_0(\eta) \geqslant h - \frac{h}{b_\lambda}\eta \quad \text{for all } 0 \leq \eta \leqslant b_0$$

and we deduce from (5.53) that

$$u_0(\eta) \geqslant W_\lambda(\eta) \quad \text{for all } 0 \leqslant \eta \leqslant b_\lambda. \tag{5.56}$$

Defining $\underline{\mathcal{W}}_\lambda = W_\lambda$ and $\underline{b}_\lambda = b_\lambda$ as in (5.16), we deduce that the pair $(\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)$ is a lower solution for Problem (5.4). □

6 A priori estimates for the solution of Problem (5.4)

6.1 A priori estimates for the solution of Problem (5.4) on the moving domain

Definition 6.1. We define

$$\underline{b}(\tau) := b\big(\tau, (\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)\big) \text{ and } \underline{W}(\eta,\tau) := W\big(\eta, \tau, (\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)\big) \text{ for all } \tau > 0, 0 \leqslant \eta \leqslant \underline{b}(\tau).$$

We start by showing successive lemmas for the function pair $(\underline{W}, \underline{b})$.

Lemma 6.2. We have the following uniform bounds in time

$$\underline{b}_\lambda \leqslant \underline{b}(\tau) := b\big(\tau, (\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)\big) \leqslant \underline{b}_\infty \leqslant \bar{b} \quad \text{for all } \ \tau \geqslant 0. \tag{6.1}$$

and

$$0 \leqslant \underline{W}(\eta,\tau) := W\big(\eta, \tau, (\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)\big) \leqslant h \text{ for all } \tau \geqslant 0, 0 \leqslant \eta \leqslant \bar{b}. \tag{6.2}$$

Proof. It follows from (5.18) and (5.23) that

$$\underline{b}_\lambda \leqslant b\big(\tau, (u_0, b_0)\big) \leqslant \bar{b} \text{ for all } \tau \geqslant 0.$$

In particular, for $(u_0, b_0) = (\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)$, we obtain

$$\underline{b}_\lambda \leqslant \underline{b}(\tau) := b\big(\tau, (\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)\big) \leqslant \bar{b} \text{ for all } \tau \geqslant 0.$$

We conclude from (5.27) that

$$\underline{b}_\lambda \leqslant \underline{b}(\tau) \leqslant \underline{b}_\infty \leqslant \bar{b},$$

then (6.1) holds.

Now we prove (6.2). Indeed, we know from (5.9) and (5.16) that $0 \leqslant \underline{\mathcal{W}}_\lambda(\eta) \leqslant h$ for all $\eta \in (0, \bar{b})$, which by Proposition 2.3 implies that

$$0 \leqslant \underline{W}(\eta,\tau) := W\big(\eta, \tau, (\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)\big) \leqslant h \quad \text{for all } \tau \geqslant 0, 0 \leqslant \eta \leqslant \bar{b},$$

so that (6.2) holds. □

Next we prove the following result.

Lemma 6.3. Let $\sigma > 0$. For all $\tau > 0$, we have that

$$\big\|\underline{W}_\eta(\cdot, \cdot + \tau)\big\|^2_{L^2(\Omega_{\sigma,\tau})} \leqslant C(\sigma), \tag{6.3}$$

for some positive constant $C(\sigma)$ which does not depend on τ and where

$$\Omega_{\sigma,\tau} := \big\{(\eta, S); 0 < \eta < \underline{b}(S + \tau), S \in (0, \sigma)\big\}. \tag{6.4}$$

Proof. We have

$$\underline{W}_\tau(\eta,\tau) = \underline{W}_{\eta\eta}(\eta,\tau) + \frac{\eta}{2}\underline{W}_\eta(\eta,\tau) \text{ for all } \tau > 0 \text{ and } 0 < \eta < \underline{b}(\tau),$$

Then,

$$\big(\underline{W} - h\big)_\tau(\eta,\tau)\big(\underline{W} - h\big)(\eta,\tau) = \underline{W}_{\eta\eta}(\eta,\tau)\big(\underline{W} - h\big)(\eta,\tau) + \frac{\eta}{2}\underline{W}_\eta(\eta,\tau)\big(\underline{W} - h\big)(\eta,\tau). \tag{6.5}$$

A direct computations yields

$$\frac{d}{d\tau}\int_0^{\underline{b}(\tau)} \big(\underline{W}(\eta,\tau) - h\big)^2 d\eta = \frac{d\underline{b}(\tau)}{d\tau}\Big(\underline{W}\big(\underline{b}(\tau),\tau\big) - h\Big)^2 + 2\int_0^{\underline{b}(\tau)} \big(\underline{W} - h\big)_\tau(\eta,\tau)\big(\underline{W} - h\big)(\eta,\tau)d\eta.$$

Since $\underline{W}\big(\underline{b}(\tau),\tau\big) = 0$, we obtain

$$\int_0^{\underline{b}(\tau)} \big(\underline{W} - h\big)_\tau(\eta,\tau)\big(\underline{W} - h\big)(\eta,\tau)d\eta = \frac{1}{2}\frac{d}{d\tau}\int_0^{b(\tau)} \big(\underline{W}(\eta,\tau) - h\big)^2 d\eta - \frac{1}{2}\frac{db(\tau)}{d\tau}h^2. \tag{6.6}$$

Then, we deduce from (6.5) and (6.6) that

$$\int_0^{\underline{b}(\tau)} \underline{W}_{\eta\eta}(\eta,\tau)\big(\underline{W} - h\big)(\eta,\tau)d\eta + \int_0^{\underline{b}(\tau)} \frac{\eta}{2}\underline{W}_\eta(\eta,\tau)\big(\underline{W} - h\big)(\eta,\tau)d\eta = \tag{6.7}$$

$$\frac{1}{2}\frac{d}{d\tau}\int_0^{b(\tau)} \big(\underline{W}(\eta,\tau) - h\big)^2 d\eta - \frac{1}{2}\frac{db(\tau)}{d\tau}h^2.$$

Next, we integrate by parts the first term on the left-hand-side of (6.7) and we use $\underline{W}\big(\underline{b}(\tau),\tau\big) = 0$ and $\underline{W}(0,\tau) = h$ to obtain

$$\int_0^{\underline{b}(\tau)} \underline{W}_{\eta\eta}(\eta,\tau)\big(\underline{W} - h\big)(\eta,\tau)d\eta = -\underline{W}_\eta\big(\underline{b}(\tau),\tau\big)h - \int_0^{\underline{b}(\tau)} |\underline{W}_\eta|^2 d\eta. \tag{6.8}$$

Due to Lemma 6.2, we have $\big|\underline{W}(\eta,\tau) - h\big| \leqslant h$ and $0 \leqslant \eta \leqslant \bar{b}$. It follows that

$$\int_0^{\underline{b}(\tau)} \frac{\eta}{2}\underline{W}_\eta(\eta,\tau)\big(\underline{W} - h\big)(\eta,\tau)d\eta \leqslant \frac{\bar{b}}{2}\, h\int_0^{\underline{b}(\tau)} |\underline{W}_\eta| d\eta. \tag{6.9}$$

Then, we deduce from (6.7), (6.8) and (6.9) that

$$\frac{1}{2}\frac{d}{d\tau}\int_0^{\underline{b}(\tau)} \big(\underline{W}(\eta,\tau) - h\big)^2 d\eta - \frac{1}{2}\frac{d\underline{b}(\tau)}{d\tau}h^2 \leqslant -\underline{W}_\eta\big(\underline{b}(\tau),\tau\big)h - \int_0^{\underline{b}(\tau)} |\underline{W}_\eta|^2 d\eta + \frac{\bar{b}}{2}\, h\int_0^{\underline{b}(\tau)} |\underline{W}_\eta| d\eta. \tag{6.10}$$

Moreover, it follows from Cauchy-Schwarz's and Young's inequalities that

$$\int_0^{\underline{b}(\tau)} |\underline{W}_\eta||1| d\eta \leqslant \frac{1}{2\varepsilon}\int_0^{\underline{b}(\tau)} |\underline{W}_\eta|^2 d\eta + \frac{\varepsilon}{2}\int_0^{\underline{b}(\tau)} |1|^2 d\eta \tag{6.11}$$

for all $\varepsilon > 0$. Since $-\underline{W}_\eta\big(\underline{b}(\tau),\tau\big) = \dfrac{d\underline{b}(\tau)}{d\tau} + \dfrac{\underline{b}(\tau)}{2}$ and in view of (6.11), (6.10) becomes

$$\frac{1}{2}\frac{d}{d\tau}\int_0^{\underline{b}(\tau)} (\underline{W} - h)^2 d\eta - \frac{h^2}{2}\frac{d\underline{b}(\tau)}{d\tau} + \int_0^{\underline{b}(\tau)} |\underline{W}_\eta|^2 d\eta \leqslant \Big(\frac{d\underline{b}(\tau)}{d\tau} + \frac{\underline{b}(\tau)}{2}\Big)h + \frac{\bar{b}\, h}{4}\Big(\frac{1}{\varepsilon}\int_0^{\underline{b}(\tau)} |\underline{W}_\eta|^2\, d\eta + \varepsilon\underline{b}(\tau)\Big). \tag{6.12}$$

Let $\sigma > 0$; we integrate both sides of the inequality (6.12) on $(\tau, \tau+\sigma)$ to obtain

$$\frac{1}{2}\int_{\tau}^{\tau+\sigma}\frac{d}{ds}\int_0^{\underline{b}(s)}(\underline{W}-h)^2\,d\eta ds - \frac{h^2}{2}\int_{\tau}^{\tau+\sigma}\frac{d\underline{b}(s)}{ds}ds + \int_{\tau}^{\tau+\sigma}\int_0^{\underline{b}(s)}|\underline{W}_\eta|^2\,d\eta ds \leqslant$$

$$\int_{\tau}^{\tau+\sigma}\left(\frac{d\underline{b}(s)}{ds}+\frac{\underline{b}(s)}{2}\right)h\,ds + \frac{\bar{b}\,h}{4}\int_{\tau}^{\tau+\sigma}\frac{1}{\varepsilon}\int_0^{\underline{b}(s)}|\underline{W}_\eta|^2\,d\eta ds + \frac{\bar{b}\,h\,\varepsilon}{4}\int_{\tau}^{\tau+\sigma}\underline{b}(s)ds.$$

Then, it follows that

$$\frac{1}{2}\int_0^{\underline{b}(\tau+\sigma)}\Big(\underline{W}(\eta,\tau+\sigma)-h\Big)^2 d\eta - \frac{1}{2}\int_0^{\underline{b}(\tau)}\Big(\underline{W}(\eta,\tau)-h\Big)^2 d\eta - \frac{h^2}{2}\Big(\underline{b}(\tau+\sigma)-\underline{b}(\tau)\Big) +$$

$$\left(1-\frac{\bar{b}\,h}{4\,\varepsilon}\right)\int_{\tau}^{\tau+\sigma}\int_0^{\underline{b}(s)}|\underline{W}_\eta|^2\,d\eta ds \leqslant \Big(\underline{b}(\tau+\sigma)-\underline{b}(\tau)\Big)h + \frac{h(2+\varepsilon\,\bar{b})}{4}\int_{\tau}^{\tau+\sigma}\underline{b}(s)ds.$$

For $\varepsilon = \dfrac{\bar{b}h}{2}$, we obtain

$$\frac{1}{2}\int_{\tau}^{\tau+\sigma}\int_0^{\underline{b}(s)}|\underline{W}_\eta|^2\,d\eta ds \;\leqslant$$

$$\frac{1}{2}\int_0^{\underline{b}(\tau)}\Big(\underline{W}(\eta,\tau)-h\Big)^2 d\eta + \left(h+\frac{h^2}{2}\right)\Big(\underline{b}(\tau+\sigma)-\underline{b}(\tau)\Big) + \frac{h(4+\bar{b}^2h)}{8}\int_{\tau}^{\tau+\sigma}\underline{b}(s)ds.$$

Since $\underline{b}(\tau) \leqslant \bar{b}$ for all $\tau > 0$ and $\big|\underline{W}(\eta,\tau)-h\big| \leqslant h$, it follows that

$$\int_{\tau}^{\tau+\sigma}\int_0^{\underline{b}(s)}|\underline{W}_\eta|^2\,d\eta ds \leqslant h^2\bar{b} + \Big(2h+h^2\Big)\bar{b} + \frac{h(4+\bar{b}^2h)}{4}\,\sigma\,\bar{b}.$$

We conclude that

$$\int_{\tau}^{\tau+\sigma}\int_0^{\underline{b}(s)}|\underline{W}_\eta|^2\,d\eta ds \leqslant C(\sigma) \text{ for some positive constant } C(\sigma). \tag{6.13}$$

Next we perform the change of variable $S = s - \tau$; then

$$\int_0^{\sigma}\int_0^{\underline{b}(S+\tau)}|\underline{W}_\eta(\eta, S+\tau)|^2\,d\eta dS \leqslant C(\sigma) \text{ for all } \tau \geqslant 0,$$

which implies that

$$\big\|\underline{W}_\eta(.,.+\tau)\big\|^2_{L^2(\Omega_{\sigma,\tau})} \leqslant C(\sigma), \tag{6.14}$$

for some positive constant $C(\sigma)$ which does not depend on τ. This completes the proof of Lemma 6.3. □

Next we show the following result.

Lemma 6.4. Let $\sigma > 0$. For all $\tau > 0$, we have that

$$\big\|\underline{W}_{\eta\eta}(.,.+\tau)\big\|^2_{L^2(\Omega_{\sigma,\tau})} \leqslant C(\sigma), \tag{6.15}$$

for some positive constant $C(\sigma)$ which does not depend on τ.

Before proving Lemma 6.4, we will recall the following result.

Lemma 6.5 (The Uniform Gronwall Lemma (Lemma 1.1 of [T, p.89]))**.** Let g and y be two positive locally integrable functions on $(0,+\infty)$ such that $\dfrac{dy}{dt}$ is locally integrable on $(0,\infty)$, which satisfy the inequalities

$$\frac{dy}{dt} \leqslant g\,y \qquad \text{for all } \ t \geqslant 0, \tag{6.16}$$

$$\int_t^{t+r} g(s)ds \leqslant a_1, \ \int_t^{t+r} y(s)ds \leqslant a_2 \ \text{ for all } \ t \geqslant 0, \tag{6.17}$$

where r, a_1, a_2, are positive constants which do not depend in t. Then

$$y(t+r) \leqslant \frac{a_2}{r}\exp(a_1), \qquad \text{for all } \ t \geqslant 0. \tag{6.18}$$

Proof of Lemma 6.4. We define

$$Z(\eta,\tau) := \underline{W}_\eta(\eta,\tau) \text{ for all } \tau > 0 \text{ and } 0 < \eta < \underline{b}(\tau), \tag{6.19}$$

where $\underline{W}(\eta,\tau)$ is defined in Definition 6.1. From Problem (5.4), we have

$$(\underline{W}_\eta)_\tau = \big(\underline{W}_\eta\big)_{\eta\eta} + \frac{\eta}{2}\big(\underline{W}_\eta\big)_\eta + \frac{\underline{W}_\eta}{2}, \quad \tau > 0, \quad 0 < \eta < \underline{b}(\tau),$$

so that

$$Z_\tau = Z_{\eta\eta} + \frac{\eta}{2}Z_\eta + \frac{Z}{2}, \quad \tau > 0, \quad 0 < \eta < \underline{b}(\tau). \tag{6.20}$$

Next we show that $Z_\eta(0,\tau) = 0$ and that $Z_\eta\big(\underline{b}(\tau),\tau\big) = Z^2\big(\underline{b}(\tau),\tau\big)$. Indeed, we have that $\big(\underline{W}(0,\tau)\big)_\tau = (h)_\tau = 0$ and $\big(\underline{W}(0,\tau)\big)_\tau = \underline{W}_\tau(0,\tau) = (\underline{W}_\eta)_\eta(0,\tau) + \frac{0}{2}\underline{W}_\eta(0,\tau)$. Then

$$Z_\eta(0,\tau) = 0. \tag{6.21}$$

Moreover, we have

$$\Big(\underline{W}\big(\underline{b}(\tau),\tau\big)\Big)_\tau = \frac{d\underline{b}(\tau)}{d\tau}\underline{W}_\eta\big(\underline{b}(\tau),\tau\big) + \underline{W}_\tau\big(\underline{b}(\tau),\tau\big) = 0, \tag{6.22}$$

and

$$\underline{W}_\tau\big(\underline{b}(\tau),\tau\big) = (\underline{W}_\eta)_\eta\big(\underline{b}(\tau),\tau\big) + \frac{\underline{b}(\tau)}{2}\underline{W}_\eta\big(\underline{b}(\tau),\tau\big). \tag{6.23}$$

We substitute (6.23) in (6.22) to obtain

$$(\underline{W}_\eta)_\eta\big(\underline{b}(\tau),\tau\big) + \frac{\underline{b}(\tau)}{2}\underline{W}_\eta\big(\underline{b}(\tau),\tau\big) + \frac{d\underline{b}(\tau)}{d\tau}\underline{W}_\eta\big(\underline{b}(\tau),\tau\big) = 0,$$

so that

$$Z_\eta\big(\underline{b}(\tau),\tau\big) + \frac{\underline{b}(\tau)}{2}Z\big(\underline{b}(\tau),\tau\big) + \frac{d\underline{b}(\tau)}{d\tau}Z\big(\underline{b}(\tau),\tau\big) = 0. \tag{6.24}$$

Since $\dfrac{d\underline{b}(\tau)}{d\tau} + \dfrac{\underline{b}(\tau)}{2} = -\underline{W}_\eta\big(\underline{b}(\tau),\tau\big) = -Z\big(\underline{b}(\tau),\tau\big)$ then (6.24) becomes

$$Z_\eta\big(\underline{b}(\tau),\tau\big) = Z^2\big(\underline{b}(\tau),\tau\big). \tag{6.25}$$

Therefore, from (6.20), (6.21) and (6.25), the time evolution Problem (5.4) leads to

$$\begin{cases} Z_\tau = Z_{\eta\eta} + \dfrac{\eta}{2} Z_\eta + \dfrac{Z}{2}, & \tau > 0, \quad 0 < \eta < \underline{b}(\tau), \\ Z_\eta(0,\tau) = 0, & \tau > 0, \\ Z_\eta(\underline{b}(\tau),\tau) = Z^2\big(\underline{b}(\tau),\tau\big), & \tau > 0, \\ \dfrac{d\underline{b}(\tau)}{d\tau} + \dfrac{\underline{b}(\tau)}{2} = -Z(\underline{b}(\tau),\tau), & \tau > 0, \\ \underline{b}(0) = \underline{b}, \\ Z(\eta,0) = \underline{W}_{\lambda,\eta}(\eta), & 0 \leqslant \eta \leqslant \underline{b}_\lambda, \end{cases} \tag{6.26}$$

where $\underline{W}_{\lambda,\eta}(\eta) = \dfrac{-h\, e^{-\frac{\lambda\eta^2}{4}}}{\int_0^{\underline{b}_\lambda} e^{-\frac{\lambda s^2}{4}} ds}$ with $\lambda \geqslant 1$. We consider the function F defined by

$$F(\tau) = \int_0^{\underline{b}(\tau)} Z^2(\eta,\tau)\, \mathrm{d}\eta. \tag{6.27}$$

Then, we compute $\dfrac{dF(\tau)}{d\tau} = \dfrac{d\underline{b}(\tau)}{d\tau}\, Z^2(\underline{b}(\tau),\tau) + 2\displaystyle\int_0^{\underline{b}(\tau)} Z_\tau(\eta,\tau)Z(\eta,\tau)\,\mathrm{d}\eta$, so that

$$\int_0^{\underline{b}(\tau)} Z_\tau(\eta,\tau)Z(\eta,\tau)\,\mathrm{d}\eta = \frac{1}{2}\frac{d}{d\tau}\int_0^{\underline{b}(\tau)} Z^2(\eta,\tau)\,\mathrm{d}\eta - \frac{1}{2}\frac{d\underline{b}(\tau)}{d\tau}\, Z^2(\underline{b}(\tau),\tau). \tag{6.28}$$

We multiply (6.20) by Z and integrate in space between 0 en $\underline{b}(\tau)$ to obtain

$$\int_0^{\underline{b}(\tau)} Z_\tau(\eta,\tau)Z(\eta,\tau)\,\mathrm{d}\eta = \int_0^{\underline{b}(\tau)} Z_{\eta\eta}(\eta,\tau)Z(\eta,\tau)\,\mathrm{d}\eta + \int_0^{\underline{b}(\tau)} \frac{\eta}{2} Z_\eta(\eta,\tau)Z(\eta,\tau)\,\mathrm{d}\eta + \int_0^{\underline{b}(\tau)} \frac{Z^2(\eta,\tau)}{2}\,\mathrm{d}\eta. \tag{6.29}$$

We integrate by parts the first term on the right-hand-side of (6.29) and using $Z_\eta(0,\tau) = 0$ and $Z_\eta(\underline{b}(\tau),\tau) = Z^2\big(\underline{b}(\tau),\tau\big)$, we deduce that

$$\int_0^{\underline{b}(\tau)} Z_{\eta\eta}(\eta,\tau)Z(\eta,\tau)\,\mathrm{d}\eta = Z^3\big(\underline{b}(\tau),\tau\big) - \int_0^{\underline{b}(\tau)} \big|Z_\eta(\eta,\tau)\big|^2\,\mathrm{d}\eta. \tag{6.30}$$

Next, since $0 \leqslant \eta \leqslant \bar{b}$, it follows that

$$\int_0^{\underline{b}(\tau)} \frac{\eta}{2} Z_\eta(\eta,\tau)Z(\eta,\tau)\,\mathrm{d}\eta \leqslant \frac{\bar{b}}{2}\int_0^{\underline{b}(\tau)} \big|Z_\eta(\eta,\tau)\big|\big|Z(\eta,\tau)\big|\,\mathrm{d}\eta. \tag{6.31}$$

Using the Cauchy-Schwarz inequality with the Young inequality, we obtain that

$$\int_0^{\underline{b}(\tau)} \frac{\eta}{2} Z_\eta(\eta,\tau)Z(\eta,\tau)\,\mathrm{d}\eta \leqslant \frac{\bar{b}}{4\varepsilon}\int_0^{\underline{b}(\tau)} |Z_\eta(\eta,\tau)|^2 d\eta + \frac{\bar{b}\varepsilon}{4}\int_0^{\underline{b}(\tau)} |Z(\eta,\tau)|^2 d\eta \tag{6.32}$$

for all $\varepsilon > 0$. Next, combining (6.28), (6.29), (6.30) and (6.32) we deduce that

$$\frac{1}{2}\frac{d}{d\tau}\int_0^{\underline{b}(\tau)} Z^2(\eta,\tau)\,\mathrm{d}\eta - \frac{1}{2}\frac{d\underline{b}(\tau)}{d\tau}\, Z^2(\underline{b}(\tau),\tau) \leqslant \tag{6.33}$$

$$Z^3\big(\underline{b}(\tau),\tau\big) - \int_0^{\underline{b}(\tau)} \big|Z_\eta(\eta,\tau)\big|^2\,\mathrm{d}\eta + \frac{\bar{b}}{4\varepsilon}\int_0^{\underline{b}(\tau)} |Z_\eta(\eta,\tau)|^2 d\eta + \frac{\bar{b}\varepsilon}{4}\int_0^{\underline{b}(\tau)} |Z(\eta,\tau)|^2 d\eta + \int_0^{\underline{b}(\tau)} \frac{Z^2(\eta,\tau)}{2}\,\mathrm{d}\eta.$$

Since $\dfrac{d\underline{b}(\tau)}{d\tau}+\dfrac{\underline{b}(\tau)}{2}=-Z(\underline{b}(\tau),\tau)$, then

$$\frac{-1}{2}\frac{d\underline{b}(\tau)}{d\tau}\,Z^2(\underline{b}(\tau),\tau)=\frac{1}{2}Z^3(\underline{b}(\tau),\tau)+\frac{\underline{b}(\tau)}{4}\,Z^2(\underline{b}(\tau),\tau).$$

So, (6.33) becomes

$$\frac{1}{2}\frac{d}{d\tau}\int_0^{\underline{b}(\tau)} Z^2(\eta,\tau)\,\mathrm{d}\eta+\frac{4\varepsilon-\bar{b}}{4\varepsilon}\int_0^{\underline{b}(\tau)}|Z_\eta(\eta,\tau)|^2 d\eta\leqslant \tag{6.34}$$
$$\frac{1}{2}Z^3(\underline{b}(\tau),\tau)-\frac{\underline{b}(\tau)}{4}\,Z^2(\underline{b}(\tau),\tau)+\frac{\bar{b}\varepsilon+2}{4}\int_0^{\underline{b}(\tau)}\big|Z(\eta,\tau)\big|^2 d\eta.$$

From (6.49) below, we have $Z\big(\underline{b}(\tau),\tau\big)=\underline{W}_\eta\big(\underline{b}(\tau),\tau\big)\leqslant 0$; setting $\varepsilon=\dfrac{\bar{b}}{2}$ then yields

$$\frac{1}{2}\frac{d}{d\tau}\int_0^{\underline{b}(\tau)} Z^2(\eta,\tau)\,\mathrm{d}\eta+\frac{1}{2}\int_0^{\underline{b}(\tau)}\big|Z_\eta(\eta,\tau)\big|^2\,\mathrm{d}\eta\leqslant\frac{\bar{b}^2+4}{8}\int_0^{\underline{b}(\tau)}\big|Z(\eta,\tau)\big|^2\,\mathrm{d}\eta. \tag{6.35}$$

It follows from (6.35), (6.13) and the uniform Gronwall Lemma 6.5 that there exists some positive constant $C(\sigma)$ which does not depend on τ such that

$$\int_0^{\underline{b}(\tau+\sigma)} Z^2(\eta,\tau+\sigma)\,\mathrm{d}\eta\leqslant C(\sigma)\ \text{ for all }\ \tau>0. \tag{6.36}$$

Next, we integrate both sides of the inequality (6.35) on $(\tau,\tau+\sigma)$ to obtain

$$\frac{1}{2}\int_\tau^{\tau+\sigma}\frac{d}{ds}\int_0^{\underline{b}(s)} Z^2(\eta,s)\;d\eta ds+\frac{1}{2}\int_\tau^{\tau+\sigma}\int_0^{\underline{b}(s)}\big|Z_\eta(\eta,s)\big|^2\,\mathrm{d}\eta\,\mathrm{d}s\leqslant\frac{\bar{b}^2+4}{8}\int_\tau^{\tau+\sigma}\int_0^{\underline{b}(s)}\big|Z(\eta,s)\big|^2\,\mathrm{d}\eta\,\mathrm{d}s. \tag{6.37}$$

Then, in view of (6.36) and the fact that $\underline{b}$ is nondecreasing, (6.37) becomes

$$\frac{1}{2}\int_0^{\underline{b}(\tau+\sigma)} Z^2(\eta,\tau+\sigma)\,\mathrm{d}\eta-\frac{1}{2}\int_0^{\underline{b}(\tau)} Z^2(\eta,\tau)\,\mathrm{d}\eta+\frac{1}{2}\int_\tau^{\tau+\sigma}\int_0^{\underline{b}(s)}\big|Z_\eta(\eta,s)\big|^2\,\mathrm{d}\eta\,\mathrm{d}s\leqslant\frac{(\bar{b}^2+4)\;\sigma\;C(\sigma)}{8}, \tag{6.38}$$

so that also

$$\int_\tau^{\tau+\sigma}\int_0^{\underline{b}(s)}\big|Z_\eta(\eta,s)\big|^2\,\mathrm{d}\eta\,\mathrm{d}s\leqslant C(\sigma)+\frac{(\bar{b}^2+4)\;\sigma\;C(\sigma)}{4}. \tag{6.39}$$

Next, we perform the change of variable $S=s-\tau$, then

$$\int_0^{\sigma}\int_0^{\underline{b}(S+\tau)}\big|Z_\eta(\eta,S+\tau)\big|^2\,\mathrm{d}\eta\,\mathrm{d}S\leqslant C(\sigma)+\frac{(\bar{b}^2+4)\;\sigma\;C(\sigma)}{4}\quad\text{for all }\tau>0$$

which implies that

$$\big\|Z_\eta(.,.+\tau)\big\|^2_{L^2(\Omega_{\sigma,\tau})}\leqslant C(\sigma)+\frac{(\bar{b}^2+4)\;\sigma\;C(\sigma)}{4}. \tag{6.40}$$

This completes the proof of Lemma 6.4. □

Next we deduce the following Corollary.

Corollary 6.6. Let $\sigma>0$. For all $\tau>0$, we have that

$$\big\|\underline{W}_\eta(.,.+\tau)\big\|_{L^2\left(0,\sigma;C^{\frac{1}{2}}(\overline{\Omega}_\tau)\right)}\leqslant C(\sigma), \tag{6.41}$$

for some positive constant $C(\sigma)$ which does not depend on τ and where

$$\Omega_\tau:=\big\{\eta;0<\eta<\underline{b}(S+\tau),S\in(0,\sigma)\big\}. \tag{6.42}$$

Proof. From Lemmas 6.3 and 6.4, we deduce that there exists some positive constant $\tilde{\tilde{C}}(\sigma)$ which does not depend on τ such that

$$\left\|\underline{W}_\eta(.,.+\tau)\right\|^2_{L^2\left(0,\sigma;H^1(\Omega_\tau)\right)} \leqslant \tilde{\tilde{C}}(\sigma). \tag{6.43}$$

Then, since $H^1(\Omega_\tau) \subset C^{\frac{1}{2}}(\overline{\Omega}_\tau)$, (6.41) follows from (6.43). □

<u>Uniform estimate of $\underline{W}(.,.+\tau)$ in $C^{\frac{1}{2},\frac{1}{4}}(\overline{\Omega}_{\sigma,\tau})$.</u>

Lemma 6.7. There exists some positive constant C which does not depend on τ such that

$$\left\|\underline{W}(.,.+\tau)\right\|_{C^{\frac{1}{2},\frac{1}{4}}(\overline{\Omega}_{\tau,\sigma})} \leqslant C, \tag{6.44}$$

where $\Omega_{\sigma,\tau} := \left\{(\eta, S); 0 < \eta < \underline{b}(S+\tau), S \in (0,\sigma)\right\}$.

Proof. There exists some positive constant $C_1(\sigma)$ which does not depend on τ such that

$$\left\|\underline{W}_\tau(.,.+\tau)\right\|_{L^2(\Omega_{\sigma,\tau})} \leqslant C_1(\sigma). \tag{6.45}$$

Indeed, we have that

$$\underline{W}_\tau(\eta,\tau) = \underline{W}_{\eta\eta}(\eta,\tau) + \frac{\eta}{2}\underline{W}_\eta(\eta,\tau) \text{ for all } \tau > 0 \text{ and } 0 < \eta < \underline{b}(\tau),$$

and from Lemmas 6.3 and 6.4, we have that

$$\left\|\underline{W}_\eta(.,.+\tau)\right\|^2_{L^2(\Omega_{\sigma,\tau})} \leqslant C_2(\sigma) \text{ for some positive constant } C_2(\sigma),$$

and

$$\left\|\underline{W}_{\eta\eta}(.,.+\tau)\right\|^2_{L^2(\Omega_{\sigma,\tau})} \leqslant C_3(\sigma) \text{ for some positive constant } C_3(\sigma).$$

Since $\eta \leqslant \bar{b}$, it follows that

$$\frac{\eta}{2}\underline{W}_\eta(.,.+\tau) \in L^2(\Omega_{\sigma,\tau}).$$

Finally, we conclude from the partial differential equation for $\underline{W}$ that the estimate (6.45) holds, so that $\underline{W}(.,.+\tau) \in \mathbb{W}_2^{2,1}(\Omega_{\sigma,\tau})$. From (Lemma 3.5 of [BHC, p.207]), we have that

$$\mathbb{W}_2^{2,1}(\Omega_{\sigma,\tau}) \subset C^{\frac{1}{2},\frac{1}{4}}(\overline{\Omega}_{\sigma,\tau}), \tag{6.46}$$

so that (6.44) holds. □

Next we show the following result.

Lemma 6.8. The function $\underline{W}_\eta$ is such that $\underline{W}_\eta(\eta,\tau) \leqslant 0$ for all $\tau > 0$ and $0 < \eta < \underline{b}(\tau)$.

Proof. We recall that $Z(\eta,\tau) := \underline{W}_\eta(\eta,\tau)$ for all $\tau > 0$ and $0 < \eta < \underline{b}(\tau)$ as defined in (6.19). From (6.26), Z satisfies the partial differential equation

$$Z_\tau = Z_{\eta\eta} + \frac{\eta}{2}Z_\eta + \frac{Z}{2}, \quad \tau > 0, \quad 0 < \eta < \underline{b}(\tau).$$

We also have that

$$Z(0,\tau) \leqslant 0 \text{ for all } \tau > 0. \tag{6.47}$$

Indeed, since $0 \leqslant \underline{W}(\eta,\tau) \leqslant h$ and $\underline{W}(0,\tau) = h$, it follows that

$$\underline{W}_\eta(0,\tau) \leqslant 0 \ \text{ for all } \tau > 0. \tag{6.48}$$

Next, we prove that

$$Z(\underline{b}(\tau),\tau) \leqslant 0. \tag{6.49}$$

Indeed, from Problem (5.4) and Lemma 5.4, we deduce that $\dfrac{d\underline{b}(\tau)}{d\tau} + \dfrac{\underline{b}(\tau)}{2} = -\underline{W}_\eta(\underline{b}(\tau),\tau)$ and $\dfrac{d\underline{b}(\tau)}{d\tau} \geqslant 0$ for all $\tau > 0$; it follows that $\underline{W}_\eta(\underline{b}(\tau),\tau) \leqslant 0$ for all $\tau > 0$. Next from (6.26), we have that

$$Z(\eta,0) = \underline{W}_{\lambda,\eta}(\eta) = \frac{-h\, e^{-\frac{\lambda\eta^2}{4}}}{\int_0^{\underline{b}_\lambda} e^{-\frac{\lambda s^2}{4}} ds} \leqslant 0 \ \text{ for } \ 0 \leqslant \eta \leqslant \underline{b}_\lambda \ \text{ with } \lambda \geqslant 1. \tag{6.50}$$

Let $T > 0$, we define

$$\mathbf{Q}_T := \{(\eta,\tau), \tau \in (0,T), 0 < \eta < \underline{b}(\tau)\}. \tag{6.51}$$

Next, we perform the change of function $Z(\eta,\tau) = \tilde{Z}(\eta,\tau)e^{\alpha\tau}$ where $\alpha > \dfrac{1}{2}$. The function $\tilde{Z}$ satisfies the equality

$$\tilde{Z}_\tau\, e^{\alpha\tau} = \tilde{Z}_{\eta\eta}\, e^{\alpha\tau} + \frac{\eta}{2}\tilde{Z}_\eta\, e^{\alpha\tau} + (\frac{1}{2} - \alpha)\tilde{Z}\, e^{\alpha\tau} \ \text{ in } \mathbf{Q}_{\mathrm{T}}, \text{ for all } \alpha > \frac{1}{2},$$

so that

$$\tilde{Z}_\tau - \left(\tilde{Z}_{\eta\eta} + \frac{\eta}{2}\tilde{Z}_\eta + (\frac{1}{2} - \alpha)\tilde{Z}\right) = 0 \ \text{ in } \ \mathbf{Q}_{\mathrm{T}}, \text{ for all } \alpha > \frac{1}{2}.$$

Now, we prove that $\tilde{Z} \leqslant 0$ in $\overline{\mathbf{Q}_T}$. Indeed, it follows from the weak maximum principle (Lemma 1 of [F3, p.34]) that $\tilde{Z}$ cannot have a positive maximum in $\mathbf{Q}_T$. Then, $\tilde{Z}$ attains its maximum on the boundary $\Gamma := \{(0,\tau), 0 \leq \tau \leq T\} \cup \{(\eta,0), 0 < \eta < \underline{b}(0)\} \cup \{(\underline{b}(\tau),\tau), 0 \leq \tau \leq T\}$.
Then, it follows from (6.47), (6.49) and (6.50) that $\tilde{Z} \leqslant 0$ on Γ, so that $\tilde{Z} \leqslant 0$ in $\overline{\mathbf{Q}_T}$ which implies that $Z \leqslant 0$ in $\overline{\mathbf{Q}_T}$. Thus, we deduce that

$$\underline{W}_\eta(\eta,\tau) \leqslant 0, \ \ \text{for all } \tau \geqslant 0, \ 0 \leqslant \eta \leqslant \underline{b}(\tau). \tag{6.52}$$

which completes the proof of Lemma 6.8. □

Lemma 6.9. Let $\tau \geqslant 0$ be arbitrary. The function $\eta \to \underline{W}_\eta(\eta,\tau)$ is nondecreasing.

Proof. To prove Lemma 6.9, we need to show that $\underline{W}_{\eta\eta}(\eta,\tau) \geqslant 0$ for each $\tau \geqslant 0$. Indeed, we define

$$G(\eta,\tau) := Z_\eta(\eta,\tau) \text{ for all } \tau > 0 \text{ and } 0 < \eta < \underline{b}(\tau).$$

We recall that $Z(\eta,\tau) := \underline{W}_\eta(\eta,\tau)$ for all $\tau > 0$ and $0 < \eta < \underline{b}(\tau)$ as defined in (6.19).

Now we derive the time evolution problem satisfied by G from the time evolution Problem (6.26) satisfied by Z. First, G satisfied the following boundary conditions

$$G(0,\tau) = 0, \quad G(\underline{b}(\tau),\tau) = Z^2\big(\underline{b}(\tau),\tau\big) \ \text{ for all } \ \tau > 0. \tag{6.53}$$

From Lemma (6.8), we have that $Z\big(\underline{b}(\tau),\tau\big) = \underline{W}_\eta\big(\underline{b}(\tau),\tau\big) \leqslant 0$ for all $\tau > 0$. It follows that

$$Z\big(\underline{b}(\tau),\tau\big) = -\sqrt{G(\underline{b}(\tau),\tau)} \ \text{ for all } \ \tau > 0. \tag{6.54}$$

Straightforward computations give

$$\begin{cases} G_\tau = G_{\eta\eta} + \dfrac{\eta}{2} G_\eta + G, & \tau > 0, \quad 0 < \eta < \underline{b}(\tau), \\ G(0,\tau) = 0, & \tau > 0, \\ G(\underline{b}(\tau),\tau) = Z^2\big(\underline{b}(\tau),\tau\big), & \tau > 0, \\ \dfrac{d\underline{b}(\tau)}{d\tau} + \dfrac{\underline{b}(\tau)}{2} = \sqrt{G(\underline{b}(\tau),\tau)}, & \tau > 0, \\ \underline{b}(0) = \underline{b}, \\ G(\eta,0) = \underline{W}_{\lambda,\eta\eta}(\eta), & 0 \leqslant \eta \leqslant \underline{b}_\lambda. \end{cases} \tag{6.55}$$

where $G(\eta,0) = \underline{W}_{\lambda,\eta\eta}(\eta) = \dfrac{\lambda\, \eta\, h\, e^{-\frac{\lambda\eta^2}{4}}}{2\int_0^{b_\lambda} e^{-\frac{\lambda s^2}{4}} ds} \geqslant 0$ with $\lambda \geqslant 1$.
Finally, we use similar arguments as in the proof of Lemma 6.8 to deduce that

$$\underline{W}_{\eta\eta}(\eta,\tau) \geqslant 0 \text{ for all } \tau \geqslant 0,\ 0 \leqslant \eta \leqslant \underline{b}(\tau). \tag{6.56}$$

This completes the proof of Lemma 6.9. □

Lemma 6.10. Let $\sigma > 0$. For all $\tau > 0$, we have that

$$\left\| \frac{d\underline{b}(.+\tau)}{d\tau} \right\|_{L^2(0,\sigma)} \leqslant C(\sigma), \tag{6.57}$$

for some positive constant $C(\sigma)$ which does not depend on τ, so that

$$\big\| \underline{b}(.+\tau) \big\|_{C^{0,\frac{1}{2}}\left([0,\sigma]\right)} \leqslant \hat{C}(\sigma), \tag{6.58}$$

for some positive constant $\hat{C}(\sigma)$ which does not depend on τ.

Proof. We only have to show (6.57). We recall that $\underline{b}(.+\tau)$ satisfies the ODE

$$\frac{d\underline{b}(.+\tau)}{d\tau} + \frac{\underline{b}(.+\tau)}{2} = -\underline{W}_\eta\big(\underline{b}(.+\tau), .+\tau\big) \quad \text{for all } \tau > 0.$$

We have that

$$\int_0^\sigma \big|\underline{W}_\eta\big(\underline{b}(s+\tau), s+\tau\big)\big|^2 ds \leqslant \int_0^\sigma \sup_{0 \leqslant \eta \leqslant \underline{b}(.+\tau)} \big|\underline{W}_\eta\big(\eta, s+\tau\big)\big|^2 ds = \big\|\underline{W}_\eta(., .+\tau)\big\|^2_{L^2(0,\sigma;C([0,\underline{b}(.+\tau)]))}. \tag{6.59}$$

From (6.59) and Corollary 6.6, we deduce that there exists some positive constant $C(\sigma)$ which does not depend on τ such that

$$\int_0^\sigma \big|\underline{W}_\eta\big(\underline{b}(s+\tau), s+\tau\big)\big|^2 ds \leqslant C(\sigma),$$

which together with Lemma 6.2 implies that (6.57) holds. □

In the following subsection, we derive estimates for the free boundary Problem (5.4) in a fixed domain.

6.2 A Priori Estimates for the solution of Problem (5.4) on the fixed domain.

It will be necessary in the sequel to reason on a fixed domain. To do so, we start by giving the transformation to the fixed domain $\hat{\Omega} := \big\{(y,\tau) \in (0,1)\times(0,\infty)\big\}$. We set

$$y = \frac{\eta}{\underline{b}(\tau)},\ \ \hat{\underline{W}}(y,\tau) = \underline{W}(\eta,\tau)\ \text{ for all }\ \tau \geqslant 0, 0 \leqslant \eta \leqslant \underline{b}(\tau). \tag{6.60}$$

Using this change of variable in the estimates obtained in Lemmas 6.3 and 6.4, with the bounds on $\underline{b}$ in Lemma 6.2, we readily get the following uniform estimates for the function $\hat{\underline{W}}$.

Lemma 6.11. Let $\sigma > 0$. For all $\tau \geqslant 0$, we have that

$$\big\|\hat{\underline{W}}_y(\cdot,\cdot+\tau)\big\|_{L^2\big((0,1)\times(0,\sigma)\big)} \leqslant C(\sigma), \tag{6.61}$$

$$\big\|\hat{\underline{W}}_{yy}(\cdot,\cdot+\tau)\big\|_{L^2\big((0,1)\times(0,\sigma)\big)} \leqslant C(\sigma), \tag{6.62}$$

for some positive constant $C(\sigma)$ which does not depend on τ.

Next, we show the following result.

Lemma 6.12. We have that

$$\hat{\underline{W}}_{yy}(y,\tau) \geqslant 0\ \text{ for all }\ \tau \geqslant 0, 0 \leqslant y \leqslant 1, \tag{6.63}$$

and the function $y \mapsto \hat{\underline{W}}_y(y,\tau)$ is nondecreasing for all $\tau \geqslant 0$. Moreover, there exists a positive constant C which does not depend on τ such that

$$\|\hat{\underline{W}}_{yy}(.,\tau)\|_{L^1(0,1)} \leqslant C\ \text{ for all }\ \tau \geqslant 0. \tag{6.64}$$

Proof. From (6.56), we have that $\underline{W}_{\eta\eta}(\eta,\tau) \geqslant 0$, for all $\tau \geqslant 0$, $0 \leqslant \eta \leqslant \underline{b}(\tau)$. Since $\underline{W}_{\eta\eta}(\eta,\tau) = \dfrac{1}{\underline{b}^2(\tau)}\hat{\underline{W}}_{yy}(y,\tau)$, we deduce that (6.63) holds. Next, we prove that

$$\underline{W}_{\eta\eta}(\eta,\tau) \text{ is uniformly bounded on }\ L^1\big([0,\underline{b}(\tau)]\big) \text{ for all }\ \tau \geqslant 0. \tag{6.65}$$

From Lemma (6.8), we have that $Z\big(\underline{b}(\tau),\tau\big) = \underline{W}_\eta\big(\underline{b}(\tau),\tau\big) \leqslant 0$ for all $\tau > 0$. Thus, we have

$$0 \leqslant \int_0^{\underline{b}(\tau)} \underline{W}_{\eta\eta}(\eta,s)\,\mathrm{d}s\ = \underline{W}_\eta(\underline{b}(\tau),\tau) - \underline{W}_\eta(0,\tau) \leqslant -\underline{W}_\eta(0,\tau). \tag{6.66}$$

We shall prove that $\underline{W}_\eta(0,\tau)$ is bounded below for $\tau \geqslant 0$. Indeed, from Lemma 5.4, we know that $\underline{W}$ is nondecreasing in time and since $\underline{W}(0,\tau) = h$ for all $\tau \geqslant 0$, it follows that

$$\underline{W}_\eta(0,0) \leqslant \underline{W}_\eta(0,\tau) \text{ for all } \tau \geqslant 0. \tag{6.67}$$

We have that $\underline{W}_\eta(0,0) = \underline{W}_{\lambda,\eta}(0) = \dfrac{-h}{\int_0^{\underline{b}_\lambda} e^{-\frac{\lambda s^2}{4}}ds}$ with $\lambda \geqslant 1$, which implies together with (6.67) that $-\underline{W}_\eta(0,\tau) \leqslant \dfrac{h}{\int_0^{\underline{b}_\lambda} e^{-\frac{\lambda s^2}{4}}ds}$, which in turn implies that

$$\int_0^{\underline{b}(\tau)} \underline{W}_{\eta\eta}(\eta,s)\,\mathrm{d}s\ \leqslant \frac{h}{\int_0^{\underline{b}_\lambda} e^{-\frac{\lambda s^2}{4}}ds}. \tag{6.68}$$

Since $\underline{W}_{\eta\eta}(\eta,s)=\dfrac{1}{\underline{b}^2(\tau)}\hat{\underline{W}}_{yy}(y,s)$ and $\underline{b}(\tau)\leqslant\bar{b}$, we deduce that

$$\int_0^1 \hat{\underline{W}}_{yy}(y,s)\,\mathrm{d}y=\|\hat{\underline{W}}_{yy}(.,s)\|_{L^1(0,1)}\leqslant\frac{\bar{b}\,h}{\int_0^{b_\lambda}e^{-\frac{\lambda s^2}{4}}ds}. \tag{6.69}$$

This complete the proof of (6.64). □

Now, we prove the following result.

Lemma 6.13. There exists a positive constant C which does not depend on τ such that

$$\|\hat{\underline{W}}_y(.,\tau)\|_{L^1(0,1)}\leqslant C\ \text{ for all }\tau\geqslant 0. \tag{6.70}$$

Proof. From Lemma 6.12, we have that the function $y\mapsto\hat{\underline{W}}_y(y,\tau)$ is nondecreasing for all $\tau\geqslant 0$. Then, it follows that

$$\|\hat{\underline{W}}_y(.,\tau)\|_{L^1(0,1)}=-\int_0^1\hat{\underline{W}}_y(y,\tau)\,dy=\hat{\underline{W}}(0,\tau)-\hat{\underline{W}}(1,\tau)=h\quad\text{ for all }\ \tau\geqslant 0. \tag{6.71}$$

Indeed, from Problem (5.4), we have that $\underline{W}(0,\tau)=h$ and $\underline{W}(\underline{b}(\tau),\tau)=0$ which implies that $\hat{\underline{W}}(0,\tau)=h$ and $\hat{\underline{W}}(1,\tau)=0$ for all $\tau\geqslant 0$. This complete the proof of Lemma 6.13. □

7 Limit Problem as $\tau\to\infty$.

Theorem 7.1. Let $(\psi,\underline{b}_\infty)$ be defined in Lemma 5.4. Then $(\psi,\underline{b}_\infty)$ is the unique stationary solution of Problem (1.12).

Before proving this theorem, we need to show some preliminary results. Let $\hat{\underline{W}}$ be defined as in (6.60). We also define

$$\hat{\psi}(y)=\psi(\eta),\quad y=\frac{\eta}{\underline{b}_\infty}\in[0,1]\ \text{ for }\ 0\leqslant\eta\leqslant\underline{b}_\infty. \tag{7.1}$$

We will derive estimates for $\hat{\psi}$. We start by showing the following result.

Lemma 7.2. We have $\hat{\psi},\hat{\psi}_y\in H^1(0,1)\subset C^{0,\frac{1}{2}}\big([0,1]\big)$.

Proof. Since $0\leqslant\underline{W}(\eta,\tau)\leqslant h$ for all $\tau\geqslant 0$ and $\eta\in[0,\underline{b}(\tau)]$, we have that

$$0\leqslant\hat{\underline{W}}(y,\tau)\leqslant h\ \text{ for all }\tau\geqslant 0,\ y\in[0,1]. \tag{7.2}$$

We deduce from (6.61) and (6.62) in Lemma 6.11 that there exists a constant $C(\sigma)>0$ such that

$$\|\hat{\underline{W}}(\cdot,\cdot+\tau)\|_{L^2(0,\sigma;H^2(0,1))}\leqslant C(\sigma) \tag{7.3}$$

for all $\tau>0$. Thus, there exists $v\in L^2\left(0,\sigma;H^2(0,1)\right)$ such that

$$\hat{\underline{W}}(\cdot,\cdot+\tau)\rightharpoonup v\ \text{ weakly in }L^2\big(0,\sigma;H^2(0,1)\big)\text{ as }\tau\to+\infty. \tag{7.4}$$

We shall prove that $v=\hat{\psi}$. First, since $\lim\limits_{\tau\to+\infty}\underline{W}(\eta,\tau)=\psi(\eta)$ for all $\eta\in\mathbb{R}^+$, it follows from (6.60) and (7.1) that

$$\lim_{\tau\to+\infty}\hat{\underline{W}}(y,\tau)=\hat{\psi}(y)\quad\text{for all }y\in[0,1], \tag{7.5}$$

and since $0 \leqslant \hat{\underline{W}} \leqslant h$, we deduce from Lebesgue Dominated Convergence Theorem that

$$\hat{\underline{W}}(\cdot, \cdot + \tau) \to \hat{\psi} \ \text{ in } L^1\left((0,1) \times (0,\sigma)\right) \ \text{as } \tau \to +\infty. \tag{7.6}$$

Using again the uniform boundedness of $\hat{\underline{W}}$ and $\hat{\psi}$, we conclude that this convergence also holds in $L^p\big((0,1) \times (0,\sigma)\big)$ for all $p \in [1,\infty)$. Hence, $v = \hat{\psi} \in H^2(0,1)$. This completes the proof of Lemma 7.2. □

Proposition 7.3. The sequence

$$\{\hat{\underline{W}}_y(\cdot,\tau)\} \text{ converges to } \hat{\psi}_y \text{ in } L^2(0,1) \ \text{ as } \tau \to +\infty. \tag{7.7}$$

Proof. From the Lemmas 6.12 and 6.13, we deduce that there exists a positive constant C independent of τ such that

$$\|\hat{\underline{W}}_y(\cdot,\tau)\|_{\mathbb{W}^{1,1}(0,1)} \leqslant C \ \text{ for all } \tau \geqslant 0. \tag{7.8}$$

The space $\mathbb{W}^{1,1}(0,1)$ is compactly embedded in $L^2(0,1)$ (see for instance the proof of Lemma 2.7 in [BGH, p.86]). Thus, it follows that there exist a subsequence $\{\hat{\underline{W}}_y(\cdot,\tau_n)\}_{n=0}^{n=\infty}$ and a function $\chi \in L^2(0,1)$ such that

$$\hat{\underline{W}}_y(\cdot,\tau_n) \to \chi \ \text{ strongly in } L^2(0,1) \ \text{ as } \tau \to \infty. \tag{7.9}$$

Now, we prove that $\chi = \hat{\psi}_y$. From (7.9), it follows that

$$\int_0^1 \hat{\underline{W}}_y(y,\tau)\varphi(y)\,dy \to \int_0^1 \chi(y)\varphi(y)\,dy \ \text{ as } \tau \to \infty \ \text{ for all } \ \varphi \in H_0^1(0,1). \tag{7.10}$$

We also have that for all $\varphi \in H_0^1(0,1)$

$$\int_0^1 \hat{\underline{W}}_y(y,\tau)\ \varphi(y)\,dy = -\int_0^1 \hat{\underline{W}}(y,\tau)\ \varphi_y(y)\,dy. \tag{7.11}$$

We then deduce from (7.6) that

$$-\int_0^1 \hat{\underline{W}}(y,\tau)\varphi_y(y)\,dy \to -\int_0^1 \hat{\psi}(y)\ \varphi_y(y)\,dy = \int_0^1 \hat{\psi}_y(y)\varphi(y)\,dy \ \text{ as } \tau \to \infty \tag{7.12}$$

for all $\varphi \in H_0^1(0,1)$. We finally deduce from (7.10), (7.11) and (7.12) that $\chi = \hat{\psi}_y$ and then (7.9) becomes

$$\hat{\underline{W}}_y(\cdot,\tau_n) \to \hat{\psi}_y \ \text{ strongly in } L^2(0,1) \text{ as } \tau \to \infty, \tag{7.13}$$

which completes the proof of Proposition 7.3. □

Next we show the following result.

Proposition 7.4 (Application of Second Dini's Theorem)**.** We have that

$$\hat{\underline{W}}_y(\cdot,\tau) \ \text{ converges uniformly to } \hat{\psi}_y \ \text{ as } \tau \to \infty \text{ on } \ [0,1]. \tag{7.14}$$

Proof. From Lemma 6.12, we have that the function $y \mapsto \hat{\underline{W}}_y(y,\tau)$ is nondecreasing for all $\tau \geqslant 0$. In view of Lemma 6.7, we recall that $\hat{\underline{W}}_y(\cdot,\tau)$ is a continuous function for all $\tau \geq 0$. From Proposition 7.3, we have that $\hat{\underline{W}}_y(.,\tau)$ converges to $\hat{\psi}_y$, as $\tau \to +\infty$, a.e. in $(0,1)$ and from Lemma 7.2, we have that $\hat{\psi}_y \in C^{0,\frac{1}{2}}([0,1])$. It follows from applying the second Dini's Theorem (Theorem 10.32 of [WMT, p. 454]) which states that "if a sequence of monotone continuous functions converges pointwise on $(0,1)$ and if the limit function is continuous in $[0,1]$, then the convergence is uniform", which completes the proof of Proposition 7.4. $\square$

Corollary 7.5. $\lim_{\tau\to+\infty} ||\hat{\underline{W}}(.,\tau) - \hat{\psi}||_{C^1([0,1])} = 0.$

Proof. It remains to show that $\hat{\underline{W}}(.,\tau)$ converges uniformly to $\hat{\psi}$ as $\tau \to \infty$. We have that

$$||\hat{\underline{W}}(.,\tau) - \hat{\psi}||_{C^0([0,1])} = \sup_{y\in[0,1]} \left| \int_0^y \hat{\underline{W}}_y(s,\tau)ds + h - \int_0^y \hat{\psi}_y(s)ds - h \right| \tag{7.15}$$

$$= \sup_{y\in[0,1]} \left| \int_0^y \hat{\underline{W}}_y(s,\tau)ds - \int_0^y \hat{\psi}_y(s)ds \right| \leqslant ||\hat{\underline{W}}_y(.,\tau) - \hat{\psi}_y||_{L^1(0,1)} \to 0 \text{ as } \tau \to \infty.$$

$\square$

Next, we prove Theorem 7.1.

Proof of Theorem 7.1. The proof will be done through successive Lemmas. The first step of the proof consists in showing the following result.

Lemma 7.6. We have $\psi(0) = h$ and $\psi(\underline{b}_\infty) = 0$.

Proof. We start by showing that $\psi(0) = h$. Indeed, we have that (recall that $\underline{W}$ is nondecreasing in time)

$$\underline{\mathcal{W}}_\lambda(\eta) = \underline{W}(\eta,0) \leqslant \underline{W}(\eta,\tau) \leqslant h. \tag{7.16}$$

Letting τ tend to $+\infty$, we deduce that

$$\underline{\mathcal{W}}_\lambda(\eta) \leqslant \psi(\eta) \leqslant h \quad \text{for all } \eta \in [0,\underline{b}_\infty].$$

Then, for $\eta = 0$, we obtain $\underline{\mathcal{W}}_\lambda(0) = h \leqslant \psi(0) \leqslant h$, that is $\psi(0) = h$.

Next, we prove that $\psi(\underline{b}_\infty) = 0$. We deduce from Corollary 7.5 that

$$\hat{\underline{W}}(1,\tau) \to \hat{\psi}(1) \text{ as } \tau \to \infty, \tag{7.17}$$

which is equivalent to

$$\underline{W}\big(\underline{b}(\tau),\tau\big) \to \psi(\underline{b}_\infty) \text{ as } \tau \to \infty. \tag{7.18}$$

Since,

$$\underline{W}\big(\underline{b}(\tau),\tau\big) = 0 \text{ for all } \tau > 0, \tag{7.19}$$

we deduce that indeed $\psi(\underline{b}_\infty) = 0$. $\square$

The following result holds.

Lemma 7.7. We have

$$\frac{\underline{b}_\infty}{2} = -\psi_\eta(\underline{b}_\infty). \tag{7.20}$$

Proof. First, we prove the corresponding relation for $\hat{\psi}_y(1)$ and then we will conclude the result for ψ_η. We recall that

$$\frac{d\underline{b}(\tau)}{d\tau} + \frac{\underline{b}(\tau)}{2} = -\underline{W}_\eta(\underline{b}(\tau), \tau) \quad \text{for all } \tau > 0. \tag{7.21}$$

In view of the change of variables (6.60) for $\hat{\underline{W}}$, the equation (7.21) becomes

$$\frac{1}{2}\frac{d\underline{b}^2(\tau)}{d\tau} + \frac{\underline{b}^2(\tau)}{2} = -\hat{\underline{W}}_y(1, \tau), \text{ for all } \tau > 0. \tag{7.22}$$

Integrating (7.22) in time between τ and $\tau+\sigma$ and performing the change of variable $S = s - \tau$, we obtain

$$\frac{1}{2}\big(\underline{b}^2(\tau+\sigma) - \underline{b}^2(\tau)\big) + \frac{1}{2}\int_0^\sigma \underline{b}^2(S+\tau)\,\mathrm{d}S = -\int_0^\sigma \hat{\underline{W}}_y(1, S+\tau)\,\mathrm{d}S. \tag{7.23}$$

Then, we deduce from Proposition 7.4 that $\hat{\underline{W}}_y(1, S+\tau)$ converges to $\hat{\psi}_y(1)$ as $\tau \to +\infty$ and recall that $\underline{b}(\tau) \to \underline{b}_\infty$ as $\tau \to +\infty$. Passing to the limit as $\tau \to +\infty$ in (7.23), we conclude that

$$\frac{\underline{b}_\infty^2}{2} = -\hat{\psi}_y(1). \tag{7.24}$$

Now, since $\psi_\eta(\eta) = \dfrac{1}{\underline{b}_\infty}\hat{\psi}_y(y),\ y = \dfrac{\eta}{\underline{b}_\infty}$ for all $0 \leqslant \eta \leqslant \underline{b}_\infty$ (see (7.1)), the relation (7.24) becomes

$$\frac{\underline{b}_\infty}{2} = -\psi_\eta(\underline{b}_\infty), \tag{7.25}$$

which completes the proof of Lemma 7.7. □

The last step of the proof of Theorem 7.1 consists in the following result.

Proposition 7.8. The function $\psi \in C^\infty([0, \underline{b}_\infty])$ and satisfies the equation

$$\psi_{\eta\eta} + \frac{\eta}{2}\psi_\eta = 0 \text{ in } (0, \underline{b}_\infty).$$

We will prove Proposition 7.8 by means of several lemmas.

Lemma 7.9. Let $\widetilde{\varphi} \in \mathcal{D}(0, \underline{b}_\infty)$ be arbitrary. Then there exists a class of functions φ satisfying the following properties

(i) $\varphi \in C^\infty\big([0, \underline{b}_\infty] \times \mathbb{R}^+\big)$,

(ii) $\varphi(0,\tau) = 0,\ \varphi\big(\underline{b}(\tau), \tau\big) = 0$ and $\varphi_\eta(0,\tau) = 0$ for all $\tau \geqslant 0$,

(iii) $\lim_{\tau\to+\infty} \varphi_\tau(\eta,\tau) = 0$ for all $\eta \in [0, \underline{b}_\infty]$,

(iv) $\lim_{\tau\to+\infty} \varphi(\eta,\tau) = \widetilde{\varphi}(\eta)$ for all $\eta \in [0, \underline{b}_\infty]$.

Proof. Let $\widetilde{\varphi} \in \mathcal{D}(0, \underline{b}_\infty)$ be given. We define the function $\varphi \in C^\infty\big([0, \underline{b}_\infty] \times \mathbb{R}^+\big)$ such that

$$\varphi(\eta,\tau) = \widetilde{\varphi}(\underline{b}_\infty\, y),\ y = \frac{\eta}{\underline{b}(\tau)} \text{ for all } \tau \geqslant 0, 0 \leqslant \eta \leqslant \underline{b}(\tau), \tag{7.26}$$

and $\varphi(\eta,\tau) = 0$ for all $\tau > 0$ and $\underline{b}(\tau) \leqslant \eta \leqslant \underline{b}_\infty$. Next, we show that φ satisfies the properties (i)-(iv).The function φ obviously satisfies (i). Property (ii) readily holds because we have $\varphi(0,\tau) =$

$\widetilde{\varphi}(0)=0$, $\varphi(\underline{b}(\tau),\tau)=\widetilde{\varphi}(\underline{b}_\infty)=0$ and $\varphi_\eta(0,\tau)=\dfrac{\underline{b}_\infty}{\underline{b}(\tau)}\,\widetilde{\varphi}_y(0)=0$. Now, we turn to (iii). We have that

$$\lim_{\tau\to+\infty}\varphi_\tau(\eta,\tau)=\lim_{\tau\to+\infty}\frac{-\eta\,\underline{b}_\infty\dfrac{d\underline{b}(\tau)}{d\tau}}{\underline{b}^2(\tau)}\,\widetilde{\varphi}_y(\underline{b}_\infty\,y). \tag{7.27}$$

Since $\dfrac{d\underline{b}(\tau)}{d\tau}+\dfrac{\underline{b}(\tau)}{2}=-\dfrac{1}{\underline{b}(\tau)}\hat{\underline{W}}_y(1,\tau)$ for all $\tau>0$, we obtain

$$\lim_{\tau\to+\infty}\frac{-\eta\,\underline{b}_\infty\dfrac{d\underline{b}(\tau)}{d\tau}}{\underline{b}^2(\tau)}\,\widetilde{\varphi}_y(\underline{b}_\infty\,y)=\lim_{\tau\to+\infty}\left(\frac{\eta\,\underline{b}_\infty}{2\underline{b}(\tau)}+\frac{\eta\,\underline{b}_\infty\hat{\underline{W}}_y(1,\tau)}{\underline{b}^3(\tau)}\right)\widetilde{\varphi}_y(\underline{b}_\infty\,y)\ \text{ for all }\ 0\leqslant\eta\leqslant\underline{b}(\tau).$$

From Proposition 7.4 and (7.24), we deduce that $\lim_{\tau\to+\infty}\hat{\underline{W}}_y(1,\tau)=\hat{\underline{\psi}}_y(1)=\dfrac{-\underline{b}_\infty^2}{2}$, which implies that

$$\lim_{\tau\to+\infty}\left(\frac{\eta\,\underline{b}_\infty}{2\underline{b}(\tau)}+\frac{\eta\,\underline{b}_\infty\hat{\underline{W}}_y(1,\tau)}{\underline{b}^3(\tau)}\right)\widetilde{\varphi}_y(\underline{b}_\infty\,y)=\left(\frac{\eta}{2}-\frac{\eta}{2}\right)\widetilde{\varphi}_y(\underline{b}_\infty\,y)=0.$$

Thus, we obtain

$$\lim_{\tau\to+\infty}\varphi_\tau(\eta,\tau)=0.$$

Finally, we show that (iv) holds; indeed we have that

$$\lim_{\tau\to+\infty}\varphi(\eta,\tau)=\lim_{\tau\to+\infty}\widetilde{\varphi}\left(\frac{\eta\,\underline{b}_\infty}{\underline{b}(\tau)}\right)=\widetilde{\varphi}(\eta).$$

This completes the proof of Lemma 7.9. □

Lemma 7.10. The function ψ satisfies

$$\int_0^{\underline{b}_\infty}\psi(\eta)\big(\widetilde{\varphi}_{\eta\eta}-\frac{\eta}{2}\widetilde{\varphi}_\eta-\frac{\widetilde{\varphi}}{2}\big)(\eta)\,\mathrm{d}\eta=0 \tag{7.28}$$

for all test functions $\widetilde{\varphi}\in\mathcal{D}(0,\underline{b}_\infty)$.

Proof. Let φ satisfying the properties $(i)-(iv)$ of Lemma 7.9 and let $\sigma>0$ be fixed. Recall that $(\underline{W},\underline{b})$ satisfies Problem 5.4, in particular we have

$$\underline{W}_\tau(\eta,\tau)=\underline{W}_{\eta\eta}(\eta,\tau)+\frac{\eta}{2}\underline{W}_\eta(\eta,\tau),\quad 0<\eta<\underline{b}(\tau),\ \tau>0. \tag{7.29}$$

By integrations by parts, we obtain

$$\int_\tau^{\tau+\sigma}\int_0^{\underline{b}(s)}\Big(\underline{W}_{\eta\eta}(\eta,s)+\frac{\eta}{2}\underline{W}_\eta(\eta,s)\Big)\,\varphi(\eta,s)\,\mathrm{d}\eta\,\mathrm{d}s$$
$$=\int_\tau^{\tau+\sigma}\int_0^{\underline{b}(s)}\underline{W}(\eta,s)\big(\varphi_{\eta\eta}-\frac{\eta}{2}\varphi_\eta-\frac{\varphi}{2}\big)(\eta,s)\,\mathrm{d}\eta\,\mathrm{d}s. \tag{7.30}$$

Moreover, we have, on the one hand

$$\int_\tau^{\tau+\sigma}\frac{d}{ds}\int_0^{\underline{b}(s)}\underline{W}(\eta,s)\varphi(\eta,s)\mathrm{d}\eta\mathrm{d}s=\int_0^{\underline{b}(\tau+\sigma)}\underline{W}(\eta,\tau+\sigma)\varphi(\eta,\tau+\sigma)\mathrm{d}\eta-\int_0^{\underline{b}(\tau)}\underline{W}(\eta,\tau)\varphi(\eta,\tau)\,\mathrm{d}\eta \tag{7.31}$$

and on the other hand,

$$\int_\tau^{\tau+\sigma}\frac{d}{ds}\int_0^{\underline{b}(s)}\underline{W}(\eta,s)\varphi(\eta,s)\mathrm{d}\eta\mathrm{d}s=\int_\tau^{\tau+\sigma}\left(\int_0^{\underline{b}(s)}\big(\underline{W}_s(\eta,s)\varphi(\eta,s)+\underline{W}(\eta,s)\varphi_s(\eta,s)\big)\mathrm{d}\eta\right.$$
$$\left.+\underline{W}(\underline{b}(s),s)\varphi(\underline{b}(s),s)\frac{d\underline{b}}{ds}(s)\right)\mathrm{d}s$$
$$=\int_\tau^{\tau+\sigma}\int_0^{\underline{b}(s)}\big(\underline{W}_s(\eta,s)\varphi(\eta,s)+\underline{W}(\eta,s)\varphi_s(\eta,s)\big)\mathrm{d}\eta\mathrm{d}s. \quad (7.32)$$

Combining (7.31) with (7.32) yields

$$\int_\tau^{\tau+\sigma}\int_0^{\underline{b}(s)}\underline{W}_s(\eta,s)\varphi(\eta,s)\mathrm{d}\eta=-\int_\tau^{\tau+\sigma}\int_0^{\underline{b}(s)}\underline{W}(\eta,s)\varphi_s(\eta,s)\mathrm{d}\eta$$
$$+\int_0^{\underline{b}(\tau+\sigma)}\underline{W}(\eta,\tau+\sigma)\varphi(\eta,\tau+\sigma)\mathrm{d}\eta-\int_0^{\underline{b}(\tau)}\underline{W}(\eta,\tau)\varphi(\eta,\tau)\,\mathrm{d}\eta. \quad (7.33)$$

We deduce from (7.29), (7.30) and (7.33) that

$$\int_0^{\underline{b}(\tau+\sigma)}\underline{W}(\eta,\tau+\sigma)\varphi(\eta,\tau+\sigma)\mathrm{d}\eta-\int_0^{\underline{b}(\tau)}\underline{W}(\eta,\tau)\varphi(\eta,\tau)\,\mathrm{d}\eta-\int_\tau^{\tau+\sigma}\int_0^{\underline{b}(s)}\underline{W}(\eta,s)\varphi_s(\eta,s)\,\mathrm{d}\eta\,\mathrm{d}s$$
$$=\int_\tau^{\tau+\sigma}\int_0^{\underline{b}(s)}\underline{W}(\eta,s)\big(\varphi_{\eta\eta}-\frac{\eta}{2}\varphi_\eta-\frac{\varphi}{2}\big)(\eta,s)\mathrm{d}\eta\mathrm{d}s. \quad (7.34)$$

Thus, since $\underline{b}(\tau)\leqslant\underline{b}_\infty$ for all $\tau\geq 0$, we can write

$$\int_0^{\underline{b}_\infty}\chi_{\left[0,\underline{b}(\tau+\sigma)\right]}\underline{W}(\eta,\tau+\sigma)\varphi(\eta,\tau+\sigma)\,\mathrm{d}\eta-\int_0^{\underline{b}_\infty}\chi_{\left[0,\underline{b}(\tau)\right]}\underline{W}(\eta,\tau)\varphi(\eta,\tau)\mathrm{d}\eta$$
$$-\int_\tau^{\tau+\sigma}\int_0^{\underline{b}(s)}\underline{W}(\eta,s)\varphi_s(\eta,s)\,\mathrm{d}\eta\,\mathrm{d}s=\int_\tau^{\tau+\sigma}\int_0^{\underline{b}(s)}\underline{W}(\eta,s)\big(\varphi_{\eta\eta}-\frac{\eta}{2}\varphi_\eta-\frac{\varphi}{2}\big)(\eta,s)\,\mathrm{d}\eta\,\mathrm{d}s. \quad (7.35)$$

Furthermore, according to Lemma 5.4, we recall that

$$\lim_{\tau\to+\infty}\underline{W}(\eta,\tau)=\psi(\eta)\ \text{ for all } 0<\eta<\underline{b}_\infty \text{ and } \lim_{\tau\to+\infty}\underline{b}(\tau)=\underline{b}_\infty.$$

Then, since φ satisfies property (iv), it follows that

$$\lim_{\tau\to+\infty}\chi_{\left[0,\underline{b}(\tau+\sigma)\right]}\underline{W}\big(\eta,\tau+\sigma\big)\varphi(\eta,\tau+\sigma)=\psi(\eta)\widetilde{\varphi}(\eta).$$

Moreover, we have

$$\left|\chi_{\left[0,\underline{b}(\tau+\sigma)\right]}\underline{W}\big(\eta,\tau+\sigma\big)\varphi(\eta,\tau+\sigma)\right|\leqslant h\,\|\varphi\|_{L^\infty(Q)}.$$

According to Lebesgue's Dominated Convergence Theorem,

$$\int_0^{\underline{b}_\infty}\chi_{\left[0,\underline{b}(\tau+\sigma)\right]}\underline{W}\big(\eta,\tau+\sigma\big)\varphi(\eta,\tau+\sigma)\,\mathrm{d}\eta\to\int_0^{\underline{b}_\infty}\psi(\eta)\widetilde{\varphi}(\eta)\,\mathrm{d}\eta\quad\text{as }\tau\to\infty. \quad (7.36)$$

Similarly, we also have that

$$\int_0^{\underline{b}_\infty}\chi_{\left[0,\underline{b}(\tau)\right]}\underline{W}\big(\eta,\tau\big)\varphi(\eta,\tau)\,\mathrm{d}\eta\to\int_0^{\underline{b}_\infty}\psi(\eta)\widetilde{\varphi}(\eta)\,\mathrm{d}\eta\quad\text{as }\tau\to\infty. \quad (7.37)$$

Now, we turn to the right-hand-side of (7.35). With the change of variables $S = s - \tau$, we obtain

$$\int_0^\sigma \int_0^{\underline{b}_\infty} \chi_{\left[0,\underline{b}(S+\tau)\right]} \underline{W}(\eta, S+\tau)\big(\varphi_{\eta\eta} - \frac{\eta}{2}\varphi_\eta - \frac{\varphi}{2}\big)(\eta, S+\tau)\,\mathrm{d}\eta\,\mathrm{d}S$$
$$\to \int_0^\sigma \int_0^{\underline{b}_\infty} \psi(\eta)\big(\widetilde{\varphi}_{\eta\eta} - \frac{\eta}{2}\widetilde{\varphi}_\eta - \frac{\widetilde{\varphi}}{2}\big)(\eta)\,\mathrm{d}\eta\,\mathrm{d}S \quad \text{as } \tau \to \infty. \tag{7.38}$$

Then, since φ satisfies the property (iii), we conclude from (7.35)–(7.38) that

$$\int_0^{\underline{b}_\infty} \psi(\eta)\big(\widetilde{\varphi}_{\eta\eta} - \frac{\eta}{2}\widetilde{\varphi}_\eta - \frac{\widetilde{\varphi}}{2}\big)(\eta)\,\mathrm{d}\eta = 0 \tag{7.39}$$

for all test functions $\widetilde{\varphi} \in \mathcal{D}(0, \underline{b}_\infty)$ which yields the result of Lemma 7.10. □

Finally, we present the proof of Lemma 7.8.

Proof of Lemma 7.8. From Lemma 7.2, we have that $\psi \in H^2(0, \underline{b}_\infty)$. Then, by means of integration by parts, we obtain

$$\int_0^{\underline{b}_\infty} \psi(\eta)\widetilde{\varphi}_{\eta\eta}(\eta)\,\mathrm{d}\eta = \int_0^{\underline{b}_\infty} \psi_{\eta\eta}(\eta)\widetilde{\varphi}(\eta)\,\mathrm{d}\eta \tag{7.40}$$

and

$$\int_0^{\underline{b}_\infty} \psi(\eta)\frac{\eta}{2}\widetilde{\varphi}_\eta(\eta)\mathrm{d}\eta = -\int_0^{\underline{b}_\infty} \Big(\psi_\eta(\eta)\frac{\eta}{2}\widetilde{\varphi}(\eta) + \frac{1}{2}\psi(\eta)\widetilde{\varphi}(\eta)\Big)\,\mathrm{d}\eta \tag{7.41}$$

for all test function $\widetilde{\varphi} \in \mathcal{D}(0, \underline{b}_\infty)$. Hence, we deduce from (7.28) that

$$\int_0^{\underline{b}_\infty} \Big(\psi_{\eta\eta}(\eta) + \frac{\eta}{2}\psi_\eta(\eta)\Big)\,\widetilde{\varphi}(\eta)\,\mathrm{d}\eta = 0, \tag{7.42}$$

for all $\widetilde{\varphi} \in \mathcal{D}(0, \underline{b}_\infty)$. This finally implies that

$$\psi \in C^\infty([0, \underline{b}_\infty]) \ \text{ and } \ \psi_{\eta\eta} + \frac{\eta}{2}\psi_\eta = 0 \ \text{ for all } \ 0 < \eta < \underline{b}_\infty. \tag{7.43}$$

This completes the proof of Lemma 7.8. □

We conclude that the pair $\Big(\underline{W}(\eta,\tau) := W\big(\eta,\tau,(\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)\big), \underline{b}(\tau) := b\big(\tau,(\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)\big)\Big)$ converges to $(\psi, \underline{b}_\infty)$ as $\tau \to \infty$. Thanks to Lemma 7.6, Lemma 7.7 and Lemma 7.8, $(\psi, \underline{b}_\infty)$ satisfies Problem (1.12) and thus $(\psi, \underline{b}_\infty)$ coincides with the unique stationary solution (U, a) of Problem (1.12). This completes the proof of Theorem 7.1. □

Similarly, one can show that $\Big(W\big(\eta,\tau,(\bar{\mathcal{W}}, \bar{b})\big), b\big(\tau,(\bar{\mathcal{W}}, \bar{b})\big)\Big)$ converges as $\tau \to \infty$ to $(\phi, \bar{b}_\infty)$ which also coincides with the unique stationary solution (U, a) of Problem (1.12). Recalling Lemma 5.7, we obtain the following result.

Theorem 7.11. Let $u_0 \in X^h(b_0) \cap \mathbb{W}^{1,\infty}\big(0, b_0\big)$ be such that $0 \leqslant u_0 \leqslant \bar{\mathcal{W}}$ in $[0, b_0]$ and $b_0 \leqslant \bar{b}$ where $(\bar{\mathcal{W}}, \bar{b})$ is defined in (5.21). Let $(W, b) = \big(W\left(\cdot,\cdot,(u_0,b_0)\right), b(\cdot,(u_0,b_0))\big)$ be the solution of Problem (5.4) with the initial data (u_0, b_0). Then

$$\lim_{\tau\to+\infty} W(\eta,\tau) = U(\eta) \ \text{ for all } \eta \in (0, a) \tag{7.44}$$

and

$$\lim_{\tau\to+\infty} b(\tau) = a \tag{7.45}$$

where (U, a) is the unique solution of the stationary Problem (1.12).

Proof. For all $\tau > 0$ and $\eta \geqslant 0$, we have that

$$W\big(\eta,\tau,(\underline{\mathcal{W}}_\lambda,\underline{b}_\lambda)\big) \leqslant W\big(\eta,\tau,(u_0,b_0)\big) \leqslant W\big(\eta,\tau,(\bar{\mathcal{W}},\bar{b})\big) \tag{7.46}$$

and

$$b\big(\tau,(\underline{\mathcal{W}}_\lambda,\underline{b}_\lambda)\big) \leqslant b\big(\tau,(u_0,b_0)\big) \leqslant b\big(\tau,(\bar{\mathcal{W}},\bar{b})\big). \tag{7.47}$$

According to Lemma 5.4 together with the fact that $(\psi,\underline{b}_\infty) = (\phi,\bar{b}_\infty) = (U,a)$, we deduce that

$$\lim_{\tau\to+\infty} W\big(\eta,\tau,(\bar{\mathcal{W}},\bar{b})\big) = \lim_{\tau\to+\infty} W(\eta,\tau,(\underline{\mathcal{W}}_\lambda,\underline{b}_\lambda)) = U(\eta), \tag{7.48}$$

$$\lim_{\tau\to+\infty} b\big(\tau,(\bar{\mathcal{W}},\bar{b})\big) = \lim_{\tau\to+\infty} b\big(\tau,(\underline{\mathcal{W}}_\lambda,\underline{b}_\lambda)\big) = a. \tag{7.49}$$

The result of Theorem 7.11 then follows from (7.46) and (7.47). □

This completes the proof of Theorem 1.1 in the introduction section.

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